Odd function: A real function $f(x)$ is said to be odd if $f(-x) = -f(x)$

If $f(x)$ is an odd function, then

$$\int_{-l}^{l} f(x)\, dx = 0$$

Even function: A real function $f(x)$ is said to be even if $f(-x) = f(x)$.

If $f(x)$ is an even function, then

$$\int_{-l}^{l} f(x)\, dx = 2\int_{0}^{l} f(x)\, dx$$

Note:

If $f(x)$ does not satisfies even and odd function then it is called neither even nor odd function.

Eg:

1. $f(x) = x^2$

$f(-x) = (-x)^2 = x^2 = f(x)$

$\Rightarrow$ $f(x)$ is an even function

2. $f(x) = x \cos x$

$f(-x) = -x \cos(-x) = -x \cos x = -f(x)$

$\Rightarrow$ $f(x)$ is an odd function

3. $f(x) = x \sin x$

$f(-x) = -x \sin(-x) = +x \sin x = f(x)$

$\Rightarrow$ $f(x)$ is an even function

4. $f(x) = |x|$

$f(-x) = |-x| = |x| = f(x)$

$\Rightarrow$ $f(x)$ is an even function

5. $f(x) = x + x^2$

$f(-x) = -x + (-x)^2 = -x + x^2$

$\Rightarrow$ $f(x)$ is neither even nor odd function.

Note:

1. For even function, $b_n = 0$
2. For odd function, $a_0 = 0$ and $a_n = 0$.
3. Even function $\times$ Even function = Even fn.
4. Odd fn. $\times$ Odd fn. = Even fn.
5. Even fn. $\times$ Odd fn. = Odd fn.
6. Odd fn. $\times$ Even fn. = Odd fn.

J. Find the Fourier series $f(x) = x$ in $(-\pi, \pi)$.

Soln.

$$f(x) = x$$
$$f(-x) = -x = -f(x)$$
$\therefore$ $f(x)$ is an odd function

$$\therefore \quad a_0 = 0 \quad \& \quad a_n = 0$$

FS $\quad f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx \quad [\because x \sin nx \rightarrow \text{even fn.}]$$

$$= \frac{2}{\pi}\left[x\left(-\frac{\cos nx}{n}\right) - 1\left(-\frac{\sin nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{x\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi \cos n\pi}{n} + 0\right]$$

$$= \frac{2}{\pi}\left(-\frac{\pi(-1)^n}{n}\right) \Rightarrow b_n = -\frac{2}{n}(-1)^n$$

$$\therefore \quad f(x) = \sum_{n=1}^{\infty} -\frac{2(-1)^n}{n} (-1)^n$$

2. Find the fourier series for the function

$f(x) = |x|$ in $(-\pi, \pi)$

Soln.

$$f(x) = |x|$$

$$f(-x) = |-x| = |x| = f(x)$$

$\therefore$ $f(x)$ is an even function

$$b_n = 0$$

FS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\, dx \qquad \because |x| \text{ is even}$$

$$|x| = x, \ (0, \pi)$$

$$= \frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{\pi}$$

$$= \frac{\pi^2}{\pi}$$

$$a_0 = \pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |x| \cos nx \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx \, dx$$

$\because$ $|x| \to$ even

$\cos nx \to$ even

even $\times$ even fn.

$\hookrightarrow$ even fn.

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$$

$u = x$ | $v = \cos nx$

$u' = 1$ | $v_1 = \frac{\sin nx}{n}$

$u'' = 0$ | $v_2 = -\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\left[(-1)^n - 1\right] \qquad \cos n\pi = (-1)^n, \quad \cos 0 = 1$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases} \qquad \begin{aligned} n=1,\ & (-1)^1 - 1 = -1-1 = -2 \\ n=2,\ & (-1)^2 - 1 = 1-1 = 0 \end{aligned}$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=\text{odd}}^{\infty} \frac{-4}{\pi n^2} \cos nx$$

---

5. Find the Fourier series for

$$f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \\ 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \end{cases}$$

Soln:

$$\phi_1(x) = 1 + \frac{2x}{\pi} \ ; \quad \phi_2(x) = 1 - \frac{2x}{\pi}$$

Now

$$\phi_1(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = \phi_2(x)$$

$\Rightarrow f(x)$ is an even function.

$$b_n = 0$$

FS $\quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right) dx$$

$$= \frac{2}{\pi}\left[x - \frac{2}{\pi}\left(\frac{x^2}{2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(\pi - \frac{\pi^2}{\pi}\right) - 0\right] = \frac{2}{\pi}[\pi - \pi]$$

$$a_0 = 0$$

Scanned with CamScanner

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} F(x) \cos nx \, dx$$

even × even → even fn

$$= \frac{2}{\pi}\int_{0}^{\pi} \left(1-\frac{2x}{\pi}\right) \cos nx \, dx$$

$u = 1-\frac{2x}{\pi}$ | $v = \cos nx$

$u' = -2/\pi$ | $v_1 = \frac{\sin nx}{n}$

$u'' = 0$ | $v_2 = \frac{-\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \left(\frac{-2}{\pi}\right)\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi}\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{2}{\pi n^2}\cos n\pi - \left(\frac{-2}{\pi}\frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{2}{\pi}\left[\frac{-2}{\pi n^2}(-1)^n + \frac{2}{\pi n^2}\right]$$

$\cos 0 = 1$

$\cos n\pi = (-1)^n$

$$= \frac{2}{\pi}\frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

$$= \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} \frac{8}{\pi^2 n^2}, & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

$n=1, \ 1-(-1)^1 = 1+1 = 2$

$n=2, \ 1-(-1)^2 = 1-1 = 0$

$$\therefore f(x) = 0/2 + \sum_{n=odd}^{\infty} \frac{8}{n^2\pi^2}\cos nx$$

$$= \sum_{n=odd}^{\infty} \frac{8}{n^2\pi^2}\cos nx$$

H.J. Find the fourier series for $f(x) = |\sin x|$, $-\pi < x < \pi$

Soln.

$$f(x) = |\sin x|$$

Now $f(-x) = |\sin(-x)|$

$$= |-\sin(x)|$$

$$= |\sin x|$$

$$= f(x)$$

$\therefore$ $f(x)$ is an even function.

$$\therefore b_n = 0$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x|\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}[-\cos x]_0^{\pi}$$

$$= -\frac{2}{\pi}[+\cos\pi - \cos 0]$$

$$= -\frac{2}{\pi}[-1-1]$$

$$= -\frac{2}{\pi}(-2)$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos nx \sin x \, dx$$

$\because 2\cos A \sin B$
$= \sin(A+B) - \sin(A-B)$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin(nx+x) - \sin(nx-x)\right] dx$$

$$= \frac{1}{\pi}\int_0^{\pi} \left[\sin(n+1)x - \sin(n-1)x\right] dx$$

$$= \frac{1}{\pi}\left[-\frac{\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right) - \left(\frac{-1}{n+1} + \frac{1}{n-1}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n+1} - \frac{(-1)^n}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[(-1)^n\left[\frac{1}{(n+1)} - \frac{1}{(n-1)}\right] + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$\cos(n+1)\pi = (-1)^{n+1}$
$= (-1)^n(-1)^1$
$= -(-1)^n$

$$= \frac{1}{\pi}\left(\frac{1}{n+1} - \frac{1}{n-1}\right)\left((-1)^n + 1\right)$$

$\cos(n-1)\pi = (-1)^{n-1}$
$= (-1)^n(-1)^{-1}$
$= -(-1)^n$

$$= \frac{1}{\pi}\left(\frac{n-1-n-1}{(n+1)(n-1)}\right)\left(1+(-1)^n\right)$$

$$= \frac{1}{\pi}\left(\frac{-2}{n^2-1}\right)\left(1+(-1)^n\right)$$

$$= -\frac{2}{\pi(n^2-1)}\left(1+(-1)^n\right)$$

$n=1,$ (

$$= \begin{cases} \frac{-4}{\pi(n^2-1)}, & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$a_1 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos x \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} |\sin x| \cos x \, dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} 2\sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \, dx \qquad \because \sin 2x = 2\sin x \cos x$$

$$= \frac{1}{\pi}\left(-\frac{\cos 2x}{2}\right)_{0}^{\pi}$$

$$= \frac{-1}{2\pi}(\cos 2\pi - \cos 0)$$

$$= \frac{-1}{2\pi}(1-1)$$

$$a_1 = 0$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{4/\pi}{2} + 0 + \sum_{n=\text{even}}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$

$$= \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=\text{even}}^{\infty} \frac{1}{n^2-1} \cos nx$$

5. Find the F.S. for $f(x) = x + x^2$ in $(-\pi, \pi)$

Soln.

$$f(x) = x + x^2$$

$$f(-x) = -x + (-x)^2 = -x + x^2 \neq f(x)$$

$$= -(x - x^2) \neq -f(x)$$

$\therefore$ $f(x)$ is neither even nor odd.

FS

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2)\, dx$$

$$= \frac{1}{\pi}\left[\frac{x^2}{2} + \frac{x^3}{3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[\left(\frac{\pi^2}{2} + \frac{\pi^3}{3}\right) - \left(\frac{\pi^2}{2} - \frac{\pi^3}{3}\right)\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\pi^3}{3}\right]$$

$$a_0 = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} (x + x^2) \cos nx\, dx$$

$u = x + x^2$	$v = \cos nx$
$u' = 1 + 2x$	$v_1 = \frac{\sin nx}{n}$
$u'' = 2$	$v_2 = -\frac{\cos nx}{n^2}$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[(x + x^2)\left(\frac{\sin nx}{n}\right) - (1 + 2x)\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$\int u v\, dx = u v_1 - u' v_2 + u'' v_3 - \ldots$$

$$= \frac{1}{\pi}\left[(x + x^2)\frac{\sin nx}{n} + (1 + 2x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{\cos n\pi}{n^2} - (1-2\pi)\frac{\cos(-n\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{(-1)^n}{n^2} - (1-2\pi)\frac{(-1)^n}{n^2}\right]$$

$\cos n\pi = (-1)^n$

$\cos(-n\pi) = \cos n\pi = (-1)^n$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}\left[1+2\pi-1+2\pi\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n^2}4\pi$$

$$a_n = \frac{4}{n^2}(-1)^n$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx \, dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} (x+x^2)\sin nx \, dx$$

$u = x+x^2$ | $v = \sin nx$

$u' = 1+2x$ | $v_1 = -\frac{\cos nx}{n}$

$u'' = 2$ | $v_2 = -\frac{\sin nx}{n^2}$

$u''' = 0$ | $v_3 = \frac{\cos nx}{n^3}$

$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - \dots$

$$= \frac{1}{\pi}\left[(x+x^2)\left(\frac{-\cos nx}{n}\right) - (1+2x)\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + (1+2x)\frac{\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_{-\pi}^{+\pi}$$

$$= \frac{1}{\pi}\left[\left(-(\pi+\pi^2)\frac{\cos n\pi}{n} + 2\frac{\cos n\pi}{n^3}\right) - \left(-(-\pi+\pi^2)\frac{\cos(-n\pi)}{n} + 2\frac{\cos(-n\pi)}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(-(\pi+\pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right) - \left((\pi-\pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{(-1)^n}{n} + 2\frac{(-1)^n}{n^3} - (\pi-\pi^2)\frac{(-1)^n}{n} - 2\frac{(-1)^n}{n^3}\right]$$

$$= \frac{1}{\pi}\frac{(-1)^n}{n}\left[-(\pi+\pi^2) - (\pi-\pi^2)\right]$$

$$= \frac{(-1)^n}{n\pi}\left[-\pi - \pi^2 - \pi + \pi^2\right]$$

$$= \frac{(-1)^n}{n\pi}(-2\pi)$$

$$b_n = -\frac{2}{n}(-1)^n$$

$$\therefore f(x) = \frac{2/3\ \pi^2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx + \sum_{n=1}^{\infty} \frac{-2}{n}(-1)^n \sin nx$$

$$= \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nx - 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx$$