Fourier Series

Periodic function:

A fn $f(x)$ is said to be periodic if $f(x+t) = f(x)$ $\forall$ real x and for some +ve no. t.

't' is known as period of $f(x)$.

Eg: $\cos x$, $\sin x$ are all periodic functions with period 2π

Dirichlet's Condition:

Any fn $f(x)$ can be developed as a Fourier series provided,

i) $f(x)$ is periodic, single valued and finite.

ii) $f(x)$ has a finite number of discontinuity in any one period.

iii) $f(x)$ has a finite number of maxima and minima.

General Fourier Series

If $f(x)$ is a periodic fn and satisfies Dirichlet's condition defined for the interval $(c, c+2l)$, then it can be represented by an finite series is called Fourier Series as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where a_0, a_n & b_n are called Fourier Co-efficients

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

1. Find the Fourier Series for the function $f(x)=x^2$ in $(0,2\pi)$

$f(x)=x^2$ in $(0,2\pi)$ $\quad c=0,\ l=\pi$

Fourier Series for the fn $f(x)$ in $[0,2\pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos nx + \sum_{n=1}^{\infty} b_n\sin nx$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\,dx = \frac{1}{\pi}\left[\frac{x^3}{3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\,\frac{8\pi^3}{3} = \frac{8\pi^2}{3}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\cos nx\,dx$$

$u = x^2$	$v = \cos nx$
$u' = 2x$	$v_1 = \frac{\sin nx}{n}$
$u'' = 2$	$v_2 = -\frac{\cos nx}{n^2}$
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{(2\pi)^2\sin n(2\pi)}{n} + \frac{2(2\pi)\cos n(2\pi)}{n^2} - \frac{2\sin n(2\pi)}{n^3}\right]$$

$$- \frac{1}{\pi}\left[0+0-\frac{2\sin 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[0 + \frac{4\pi}{n^2} - 0 - 0 - 0 + 0\right]$$

$$= \frac{4}{n^2}$$

To find b_n:

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2 \sin nx\,dx$$

$u = x^2$	$V = \sin nx$
$u' = 2x$	$V_1 = -\frac{\cos nx}{n}$
$u'' = 2$	$V_2 = -\frac{\sin nx}{n^2}$
$u''' = 0$	$V_3 = \frac{\cos nx}{n^2}$

$$= \frac{1}{\pi}\left[-x^2\frac{\cos nx}{n} + \frac{2x\sin nx}{n^2} + 2\frac{\cos nx}{n^2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(2\pi)^2\frac{\cos n(2\pi)}{n} + \frac{2(2\pi)\sin n(2\pi)}{n^2} + \frac{2\cos n(2\pi)}{n^2}\right]$$

$$-\frac{1}{\pi}\left[0+0+2\frac{\cos 0}{n^2}\right]$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2}{n} + 0 + \frac{2}{n^2} - \frac{2}{n^2}\right] = -\frac{4\pi}{n}$$

$$\therefore f(x) = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}\cos nx - \sum_{n=1}^{\infty}\frac{4\pi}{n}\sin nx.$$

Interval: $[0 \text{ to } 2l)$

3. Find the Fourier Series for the fn $f(x) = x^2$ in $[0, 2l)$

$f(x) = x^2$ in $[0, 2l)$

The Fourier Series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{2l} x^2\,dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l} = \frac{1}{l}\,\frac{8l^3}{3} = \frac{8l^2}{3}$$

To find a_n:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_0^{2l} x^2 \cos\frac{n\pi x}{l}\,dx$$

$u = x^2$	$v = \cos\frac{n\pi x}{l}$
$u' = 2x$	$v_1 = \frac{\sin\frac{n\pi x}{l}}{n\pi/l}$
$u'' = 2$	$v_2 = -\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^2}$
$u''' = 0$	$v_3 = -\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)^3}$

$$a_n = \frac{1}{l}\left[x^2\frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)} + 2x\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right] - 2\,\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[0 + 4l\,\frac{1}{\frac{n^2\pi^2}{l^2}} - 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{l}\,\frac{4l^3}{n^2\pi^2} = \frac{4l^2}{n^2\pi^2}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l} x^2 \sin\left(\frac{n\pi x}{l}\right)dx$$

$u = x^2$

$u' = 2x$

$u'' = 2$

$u''' = 0$

$v = \sin\frac{n\pi x}{l}$

$v_1 = -\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$v_2 = -\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$b_n = \frac{1}{l}\left[-x^2\frac{\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + 2x\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} + 2\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{-(2l)^2}{\frac{n\pi}{l}} + 0 + \frac{2}{\frac{n^3\pi^3}{l^3}} - 0 + 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left(\frac{-4l^2 l}{n\pi}\right) = \frac{-4l^2}{n\pi}$$

The Fourier Series is

$$f(x) = \frac{\frac{8l^2}{3}}{2} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\left(-\frac{4l^2}{n\pi}\right)\sin\frac{n\pi x}{l}$$

$$= \frac{4l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi}\cos\left(\frac{n\pi x}{l}\right) - \sum_{n=1}^{\infty}\frac{4l^2}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

3. Find the Fourier series for $f(x) = (\pi - x)^2$ in $(0, 2\pi)$

Soln.

Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2\, dx$$

$$= \frac{1}{\pi}\left[\frac{(\pi - x)^3}{-3}\right]_0^{2\pi} \qquad \because \int x^n\, dx = \frac{x^{n+1}}{n+1}$$

$$= \frac{-1}{3\pi}\left[(\pi - 2\pi)^3 - (\pi - 0)^3\right]$$

$$= \frac{-1}{3\pi}\left[(-\pi)^3 - \pi^3\right]$$

$$= -\frac{1}{3\pi}\left[-\pi^3 - \pi^3\right]$$

$$= -\frac{1}{3\pi}(-2\pi^3)$$

$$a_0 = \frac{2}{3}\pi^2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(x) \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} (\pi - x)^2 \cos nx\, dx$$

$u = (\pi - x)^2$	$v = \cos nx$
$u' = 2(\pi - x)(-1)$	$v_1 = \frac{\sin nx}{n}$
$= -2(\pi - x)$	$v_2 = -\frac{\cos nx}{n^2}$
$u'' = -2(-1) = 2$	
$u''' = 0$	$v_3 = -\frac{\sin nx}{n^3}$

By Bernoulli's formula,

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - (-2(\pi-x))\left(-\frac{\cos nx}{n^2}\right) + 2\left(-\frac{\sin nx}{n^3}\right)\right]_{x=0}^{2\pi}$$

$$= \frac{1}{\pi}\left[\underbrace{(\pi-x)^2\frac{\sin nx}{n}}_{\sin 2n\pi\to 0} - 2(\pi-x)\left(\frac{\cos nx}{n^2}\right) - \underbrace{2\left(\frac{\sin nx}{n^3}\right)}_{\sin 2n\pi\to 0}\right]_{x=0}^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-2(\pi-2\pi)\frac{\cos 2n\pi}{n^2}\right) - \left(-2(\pi-0)\frac{\cos 0}{n^2}\right)\right]$$

$$= \frac{1}{\pi}\left[-2(-\pi)\frac{1}{n^2} + 2\pi\frac{1}{n^2}\right]$$

$\left[\because \cos 2n\pi = 1,\ \sin 2n\pi = 0,\ \cos 0 = 1,\ \sin 0 = 0\right]$

$$= \frac{1}{\pi}\left[\frac{2\pi}{n^2} + \frac{2\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$a_n = \frac{4}{n^2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx$$

$u = (\pi-x)^2$	$v = \sin nx$
$u' = 2(\pi-x)(-1) = -2(\pi-x)$	$v_1 = -\frac{\cos nx}{n}$
$u'' = -2(-1) = 2$	$v_2 = -\frac{\sin nx}{n^2}$
$u''' = 0$	$v_3 = \frac{\cos nx}{n^3}$

By Bernoulli's formula

$$\int uv\,dx = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \cdots$$

$$= \frac{1}{\pi}\left[(\pi-x)^2\left(-\frac{\cos nx}{n}\right) - (-2(\pi-x))\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-(\pi-x)^2\frac{\cos nx}{n} - \underbrace{2(\pi-x)\frac{\sin nx}{n^2}}_{\sin 2n\pi\to 0} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= +\frac{1}{\pi}\left[\left(-(\pi-2\pi)^2\frac{\cos 2n\pi}{n}+2\frac{\cos 2n\pi}{n^3}\right)-\right.$$

$$\left.\left(-(\pi-0)^2\frac{\cos 0}{n}+2\frac{\cos 0}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)-\left(-\frac{\pi^2}{n}+\frac{2}{n^3}\right)\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi^2}{n}+\frac{2}{n^3}+\frac{\pi^2}{n}-\frac{2}{n^3}\right]$$

$b_n = 0$

$$\therefore \quad f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \frac{2/3\,\pi^2}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + 0$$

$$f(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{1}{n^2}\cos nx$$

4]. Find the Fourier series for $f(x) = (l-x)^2$ in $(0, 2l)$.

Soln.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2\,dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_{x=0}^{2l}$$

$$= \frac{-1}{3l}\left[(l-2l)^3-(l-0)^3\right] = \frac{-1}{3l}\left[-l^3-l^3\right] = \frac{-(-2l^3)}{3l}$$

$a_0 = 2/3\, l^2$

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$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l}(l-x)^2\cos\left(\frac{n\pi x}{l}\right)dx$$

$u=(l-x)^2$	$v=\cos\left(\frac{n\pi x}{l}\right)$
$u'=2(l-x)(-1)$ $=-2(l-x)$	$v_1=\frac{\sin\left(\frac{n\pi x}{l}\right)}{\frac{n\pi}{l}}$
$u''=-2(-1)=2$	$v_2=\frac{-\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^2}$
$u'''=0$	$v_3=\frac{-\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-(-2(l-x))\left(\frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right)+2\left(-\frac{\sin\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_{x=0}^{2l}$$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right)-2(l-x)\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}-2\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$\sin 2n\pi \to 0$ $\sin 0$ $\qquad$ $\sin 2n\pi \to 0$ $\sin 0$

$$= \frac{1}{l}\left[-2(l-2l)\frac{l^2}{n^2\pi^2}\cos 2n\pi-\left(-2(l-0)\frac{l^2}{n^2\pi^2}\cos 0\right)\right]$$

$$= \frac{1}{l}\left[2l\frac{l^2}{n^2\pi^2}+2l\frac{l^2}{n^2\pi^2}\right]$$

$$= \frac{1}{l}\left[\frac{2l^3}{n^2\pi^2}+\frac{2l^3}{n^2\pi^2}\right]$$

$$= \frac{1}{l}\frac{4l^3}{n^2\pi^2}$$

$$a_n = \frac{4l^2}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2 \sin\frac{n\pi x}{l}\,dx$$

$u = (l-x)^2$

$u' = 2(l-x)(-1)$

$= -2(l-x)$

$u'' = -2(-1)$

$= 2$

$u''' = 0$

$v = \sin\frac{n\pi x}{l}$

$v_1 = \frac{-\cos\frac{n\pi x}{l}}{n\pi/l}$

$v_2 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$v_3 = \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}$

$$= \frac{1}{l}\left[(l-x)^2\left(\frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}\right) - \left(-2(l-x)\right)\left(\frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right) + 2\left(\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^{2l}$$

$$= \frac{1}{l}\left[-(l-x)^2 \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)} - \cancel{2(l-x)\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}} + 2\left(\frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}\right)\right]_0^{2l}$$

$\sin 2n\pi \to 0$

$\sin 0 \to 0$

$$= \frac{-1}{l}\left[\left(-(l-2l)^2\frac{l}{n\pi}\cos 2n\pi + 2\frac{l^3}{n^3\pi^3}\cos 2n\pi\right) - \left(-(l-0)^2\frac{l}{n\pi}\cos 0 + 2\frac{l^3}{n^3\pi^3}\cos 0\right)\right]$$

$$= \frac{-1}{l}\left[-l^2\frac{l}{n\pi} + \frac{2l^3}{n^3\pi^3} + l^2\frac{l}{n\pi} + \frac{2l^3}{n^3\pi^3}\right]$$

$b_n = 0$

$$\therefore\ f(x) = a_0/2 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} \underbrace{b_n}_{0} \sin\frac{n\pi x}{l}$$

$$f(x) = l^2/3 + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}$$

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