Harmonic Analysis.

The process of finding the Fourier Series for a function given by numerical values is known as Harmonic analysis.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right)$$

$$a_n = 2\left(\frac{\Sigma y \cos nx}{N}\right) \quad \& \quad b_n = 2\left(\frac{\Sigma y \sin nx}{N}\right)$$

Fundamental (or) First Harmonic function:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

Second Harmonic function:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x.$$

Third Harmonic function:

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

There are four types:

Type 1: The value of x given in terms of π

Type 2: The value of x given in terms of degrees.

Type 3: The value of x given in terms of time function T.

Type 4: The value of x given in numerical values.

Type 1: The value of x given in terms of π

1) Find the Fourier series expansion of period 2π for $y = f(x)$ defined in $(0, 2\pi)$ by means of the values given below, [upto second harmonic]

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1.0	1.4	1.9	1.7	1.5	1.2	1.0

$N = 7$

If the function value of I & last ordinates coincide we can omit any one of them.

$\therefore N = 6$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

x	$y = f(x)$	$y \cos x$	$y \cos 2x$	$y \sin x$	$y \sin 2x$
0	1.0	1.0	1.0	0	0
$\pi/3$	1.4	0.7	-0.7	1.212	1.212
$2\pi/3$	1.9	-0.95	-0.95	1.645	-1.645
π	1.7	-1.7	1.7	0	0
$4\pi/3$	1.5	-0.75	-0.75	-1.299	1.299
$5\pi/3$	1.2	0.6	-0.6	-1.039	-1.039
	$\Sigma y = 8.7$	$\Sigma y \cos x = -1.1$	$\Sigma y \cos 2x = -0.3$	$\Sigma y \sin x = 0.519$	$\Sigma y \sin 2x = -0.1732$

$$a_0 = 2\left(\frac{\Sigma y}{N}\right) = 2\,\frac{8.7}{6} = 2.9$$

$$a_1 = \frac{2\,\Sigma y \cos x}{N} = \frac{2(-1.1)}{6} = -0.367$$

$$a_2 = \frac{2\,\Sigma y \cos 2x}{N} = \frac{2(-0.3)}{6} = -0.1$$

$$b_1 = \frac{2\,\Sigma y \sin x}{N} = \frac{2(0.519)}{6} = 0.173$$

$$b_2 = \frac{2\,\Sigma y \sin 2x}{N} = \frac{2(-0.1732)}{6} = -0.057$$

$f(x) = 1.45 - 0.367\cos x - 0.1\cos 2x + 0.173\sin x - 0.057\sin 2x$

2. Find the Fourier Series as far as the II harmonic to represent the function given in the following data.

x	0	1	2	3	4	5
y	9	18	24	28	26	20

Here $N = 6$

$2l = 6 \Rightarrow l = 3$.

$\therefore$ The Fourier Series is

$$f(x) = \frac{a_0}{2} + a_1\cos\frac{1\pi x}{l} + a_2\cos\frac{2\pi x}{l} + b_1\sin\frac{1\pi x}{l} + b_2\sin\frac{2\pi x}{l}$$

$$= \frac{a_0}{2} + a_1\cos\frac{\pi x}{3} + a_2\cos\frac{2\pi x}{3} + b_1\sin\frac{\pi x}{3} + b_2\sin\frac{2\pi x}{3}$$

x	$y = f(x)$	$y\cos\frac{\pi x}{3}$	$y\cos\frac{2\pi x}{3}$	$y\sin\frac{\pi x}{3}$	$y\sin\frac{2\pi x}{3}$
0	9	9	9	0	0
1	18	9	-9	15.7	15.6
2	24	-12	-24	20.9	0
3	28	-28	28	0	0
4	26	-13	-13	-22.6	22.6
5	20	10	10	-17.4	-17.4
	$\Sigma y = 125$	$\Sigma y\cos\frac{\pi x}{3} = -25$	$\Sigma y\cos\frac{2\pi x}{3} = -19$	$\Sigma y\sin\frac{\pi x}{3} = -3.4$	$\Sigma y\sin\frac{2\pi x}{3} = 20.8$

$$a_0 = 2\frac{\Sigma y}{N} = 2\left(\frac{125}{6}\right) = 41.66$$

$$a_1 = 2\frac{\Sigma y\cos\frac{\pi x}{3}}{N} = 2\left(\frac{-25}{6}\right) = -8.33$$

$$a_2 = 2\frac{\Sigma y\cos\frac{2\pi x}{3}}{N} = 2\left(-\frac{19}{6}\right) = -6.33$$

$$b_1 = 2\frac{\Sigma y \sin \frac{\pi x}{3}}{N} = 2\left(\frac{-3.4}{6}\right) = -1.13$$

$$b_2 = 2\frac{\Sigma y \sin \frac{2\pi x}{3}}{N} = 2\left(\frac{20.8}{6}\right) = 6.9$$

$$\therefore f(x) = 20.83 - 8.33\cos\frac{\pi x}{3} - 6.33\cos\frac{2\pi x}{3} - 1.13\sin\frac{\pi x}{3} + 6.9\sin\frac{2\pi x}{3}$$

3. Find the Fourier Series expansion defined in $(0, T)$ by means of the table values given below. Find the series upto the 2nd harmonic

t sec	0	T/6	T/3	T/2	2T/3	5T/6	T
A temp	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

$\therefore N = 6$, $T = 2\pi = 360^\circ$

$2l = 2\pi \Rightarrow l = \pi$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$\therefore f(x) = \frac{a_0}{2} + a_1\cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

Sub $T = 2\pi$ in the table.

t sec	0	60°	120°	180°	240°	300°
A temp	1.98	1.30	1.05	1.30	-0.88	-0.25

x	$y=f(x)$	$y\cos x$	$y\cos 2x$	$y\sin x$	$y\sin 2x$
0	1.98	1.98	1.98	0	0
60°	1.30	0.65	-0.65	1.126	1.126
120°	1.05	-0.525	-0.525	0.909	-0.909
180°	1.30	-1.3	1.3	0	0
240°	-0.88	0.44	0.44	0.762	-0.762
300°	-0.25	-0.125	0.125	0.217	0.217
	$\Sigma y = 4.5$	$\Sigma y\cos x = 1.12$	$\Sigma y\cos 2x = 2.67$	$\Sigma y\sin x = 3.014$	$\Sigma y\sin 2x = -0.328$

$$a_0 = \frac{2\Sigma y}{N} = 2\left(\frac{4.5}{6}\right) = 1.5$$

$$a_1 = 2\frac{\Sigma y \cos x}{N} = 2\left(\frac{1.12}{6}\right) = 0.373$$

$$a_2 = 2\frac{\Sigma y \cos 2x}{N} = 2\left(\frac{2.67}{6}\right) = 0.89$$

$$b_1 = 2\left(\frac{\Sigma y \sin x}{N}\right) = 2\left(\frac{3.014}{6}\right) = 1.005$$

$$b_2 = 2\left(\frac{\Sigma y \sin 2x}{N}\right) = 2\left(\frac{-0.328}{6}\right) = -0.109$$

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \sin 2x$$

$$= 0.75 + 0.373 \cos x + 0.89 \cos 2x + 1.005 \sin x - 0.109 \sin 2x$$