Parseval's Identity.

Let $f(x)$ be a periodic function defined in the interval $(-l, l)$ then

$$\frac{1}{2l}\int_{-l}^{l} [f(x)]^2 \, dx = \left[\frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)\right]$$

for the interval $(0, 2l)$

$$\frac{1}{2l}\int_{0}^{2l} [f(x)]^2 \, dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

1) Find the Fourier Series of $f(x) = x^2$ in $-\pi < x < \pi$ & deduce the

i) $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}$

ii) $\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \cdots = \frac{\pi^2}{12}$

iii) $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$

iv) $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots = \frac{\pi^4}{90}$

$f(x) = x^2$ in $-\pi < x < \pi$

$f(x)$ is even function

$\therefore \quad b_n = 0$

$$\therefore f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2\,dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\left[\frac{\pi^3}{3}\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x^2 \cos nx\,dx$$

$u = x^2$, $u' = 2x$, $u'' = 2$, $u''' = 0$

$v = \cos nx$, $v_1 = \frac{\sin nx}{n}$, $v_2 = -\frac{\cos nx}{n^2}$, $v_3 = -\frac{\sin nx}{n^3}$

$$= \frac{2}{\pi}\left[x^2 \frac{\sin nx}{n} + 2x\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi^2 \frac{\sin n\pi}{n} + 2\pi\frac{\cos n\pi}{n^2} - 2\frac{\sin n\pi}{n^3} - 0 - 0 + \frac{2\sin 0}{n^3}\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{2\pi(-1)^n}{n^2} - 0 + 0\right]$$

$$= \frac{4}{n^2}(-1)^n$$

$\therefore$ The Fourier Series is

$$f(x) = \frac{\frac{2\pi^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2}(-1)^n \cos nx$$

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}\cos nx \quad \text{— } (1)$$

ii) Put $x=0$ in (1)

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(1)$$

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots$$

$$\therefore \frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots \quad -(2)$$

i) Put $x=\pi$ in (1)

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos n\pi$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}(-1)^n$$

$$\frac{3\pi^2-\pi^2}{3} = 4\left[\frac{(-1)(-1)}{1^2} + \frac{1}{2^2} + \frac{(-1)(-1)}{3^2} + \cdots\right]$$

$$\frac{2\pi^2}{12} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\therefore \frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots \quad -(3)$$

iii) Adding (2) & (3)

$$\frac{\pi^2}{12} + \frac{\pi^2}{6} = \left[\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots\right] + \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2 + 2\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{3\pi^2}{2\times 12} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\therefore \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

iv) By using Parseval's identity,

$$\int_{-\pi}^{\pi} [f(x)]^2 \, dx = 2\pi \left[\frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \right]$$

$$\int_{-\pi}^{\pi} (x^2)^2 \, dx = 2\pi \left[\frac{\left(\frac{2\pi^2}{3}\right)^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} \left[\frac{4}{n^2} (-1)^n \right]^2 + 0 \right]$$

$$\int_{-\pi}^{\pi} x^4 \, dx = 2\pi \left[\frac{\pi^4}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4} (-1)^{2n} \right]$$

$$\left[\frac{x^5}{5} \right]_{-\pi}^{\pi} = 2\pi \left[\frac{\pi^4}{9} + \frac{1}{2} \sum_{n=1}^{\infty} \frac{16}{n^4} \right]$$

$$\frac{\pi^5}{5} + \frac{\pi^5}{5} = \frac{2\pi^5}{9} + \frac{2\pi}{2} \times 16 \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \right]$$

$$\frac{2\pi^5}{5} - \frac{2\pi^5}{9} = 16\pi \left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots \right]$$

$$\frac{8\pi^5}{45} \times \frac{1}{16\pi} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$

$$\therefore \frac{\pi^4}{90} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \cdots$$