1. Obtain the Fourier expansion of $x \sin x$ as a cosine series in $[0, \pi)$ and hence deduce the value of

$$1 + \frac{2}{1\cdot 3} - \frac{2}{3\cdot 5} + \frac{2}{5\cdot 7} - \cdots$$

The Fourier cosine series of $f(x)$ in $[0, \pi)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

To find a_0:

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \sin x \, dx$$

$u = x$	$v = \sin x$
$u' = 1$	$v_1 = -\cos x$
$u'' = 0$	$v_2 = -\sin x$

$$a_0 = \frac{2}{\pi}\left[-x\cos x + \sin x\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\pi\cos\pi + \sin\pi + 0 - \sin 0\right]$$

$$= \frac{2}{\pi}[\pi] = 2.$$

To find a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx$$

$$= \frac{2}{\pi}\int_0^{\pi} x \sin x \cos nx \, dx.$$

$$\sin x \cos nx = \frac{1}{2}\left[\sin(1+n)x + \sin(1-n)x\right]$$

$$= \frac{1}{2}\frac{2}{\pi}\int_0^{\pi} x \sin(1+n)x \, dx + \int_0^{\pi} x \sin(1-n)x \, dx$$

$u = x$	$v = \sin(1+n)x$	$u = x$	$v = \sin(1-n)x$
$u' = 1$	$v_1 = -\frac{\cos(1+n)x}{1+n}$	$u' = 1$	$v_1 = -\frac{\cos(1-n)x}{1-n}$
$u'' = 0$	$v_2 = -\frac{\sin(1+n)x}{(1+n)^2}$	$u'' = 0$	$v_2 = -\frac{\sin(1-n)x}{(1-n)^2}$

$$= \frac{1}{\pi}\left[\left(-x\frac{\cos(1+n)x}{1+n} + \frac{\sin(1+n)x}{(1+n)^2}\right)_0^\pi + \left(\frac{-x\cos(1-n)x}{1-n} + \frac{\sin(1-n)x}{(1-n)^2}\right)_0^\pi\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos(1+n)\pi}{1+n} - \frac{\pi \cos(1-n)\pi}{1-n}\right]$$

$$= -\left[\frac{\cos(1+n)\pi}{1+n} - \frac{\cos(1-n)\pi}{1-n}\right]$$

$$= -\left[\frac{\cos\pi\cos n\pi - \sin\pi\sin n\pi}{1+n} - \frac{\cos\pi\cos n\pi + \sin\pi\sin n\pi}{1-n}\right]$$

$$= \frac{-(-1)(-1)^n + 0}{1+n} + \frac{(-1)(-1)^n - 0}{1-n}$$

$$= \frac{(-1)^n}{1+n} + \frac{(-1)^n}{1-n} = (-1)^n\left[\frac{1-n+1+n}{1-n^2}\right]$$

$$= \frac{2(-1)^n}{1-n^2}, \quad \text{provided } n \neq 1$$

When $n=1$,

$$a_1 = \frac{2}{\pi}\int_0^\pi x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^\pi x \sin 2x \, dx$$

$u = x$	$v = \sin 2x$
$u' = 1$	$v_1 = -\frac{\cos 2x}{2}$
$u'' = 0$	$v_2 = -\frac{\sin 2x}{4}$

$$a_1 = \frac{1}{\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^\pi$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos 2\pi}{2} + \frac{\sin 2\pi}{4} - 0 + 0\right]$$

$$a_1 = \frac{-1}{2}$$

$\therefore$ The Fourier Cosine Series is

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$= \frac{2}{2} - \frac{1}{2}\cos x + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{1-n^2} \cos nx .$$

$$x \sin x = 1 - \frac{\cos x}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{1-n^2} \cos nx$$

Deduction:

Put $x = \pi/2$

$$\frac{\pi}{2} \sin \frac{\pi}{2} = 1 - \frac{\cos \frac{\pi}{2}}{2} + 2\sum_{n=2}^{\infty} \frac{(-1)^n}{1-n^2} \cos n\frac{\pi}{2}$$

$$\frac{\pi}{2} = 1 - 0 + 2\sum_{n=2,4,\dots} \frac{(-1)^n}{(1-n)(1+n)} \cos \frac{n\pi}{2}$$

$$\frac{\pi}{2} = 1 + 2\frac{1}{-3}\cos \pi + 2\frac{1}{(-3)(5)}\cos 2\pi + 2\frac{1}{(-5)(7)}\cos 3\pi + \dots$$

$$\frac{\pi}{2} = 1 + 2\left[\frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots\right]$$

$$\frac{\frac{\pi}{2} - 1}{2} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$$

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} - \dots$$

$$\therefore \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7} \dots = \frac{\pi - 2}{4}$$

3. Find the half range cosine series for $f(x) = x$ in $0 < x < \pi$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} x \, dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\frac{\pi^2}{2} = \pi$$

$$\therefore a_0 = \pi$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$u = x$	$v = \cos nx$
$u' = 1$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = \frac{-\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\left[x\frac{\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi\frac{\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - 0 - \frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[0 + \frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$= \frac{-2}{n^2\pi}\left[1 - (-1)^n\right]$$

$$a_n = \begin{cases} \frac{-4}{n^2\pi}, & n \text{ is even} \\ 0, & n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$$

Half range cosine series:

$$f(x) = x(\pi - x) \quad \text{in} \quad 0 < x < \pi$$

$$a_0 = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2)\, dx.$$

$$= \frac{2}{\pi} \left[\frac{x^2 \pi}{2} - \frac{x^3}{3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[\frac{\pi^3}{2} - \frac{\pi^3}{3} \right] = \frac{2}{\pi} \left[\frac{\pi^3}{6} \right] = \frac{\pi^2}{3}.$$

$$a_n = \frac{2}{\pi} \int_0^{\pi} (x\pi - x^2) \cos nx \, dx$$

$$= \frac{2}{\pi} \quad \begin{array}{l|l} u = x\pi - x^2 & v = \cos nx \\ u' = \pi - 2x & v_1 = \frac{\sin nx}{n} \\ u'' = -2 & v_2 = -\frac{\cos nx}{n^2} \\ u''' = 0 & v_3 = -\frac{\sin nx}{n^3} \end{array}$$

$$a_n = \frac{2}{\pi} \left[(x\pi - x^2) \frac{\sin nx}{n} + (\pi - 2x) \frac{\cos nx}{n^2} + 2 \frac{\sin nx}{n^3} \right]_0^{\pi}$$

$$= \frac{2}{\pi} \left[0 - \pi \frac{\cos n\pi}{n^2} + \frac{2 \sin n\pi}{n^3} - 0 - \pi \frac{\cos 0}{n^2} - \frac{2 \sin 0}{n^3} \right]$$

$$= \frac{2}{\pi} \left[-\pi \frac{(-1)^n}{n^2} + 0 - \frac{\pi}{n^2} - 0 \right]$$

$$= -\frac{2}{\pi} \frac{\pi}{n^2} \left[1 + (-1)^n \right]$$

$$= -\frac{2}{n^2} \left[1 + (-1)^n \right]$$

$$\therefore a_n = \begin{cases} -\frac{4}{n^2} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore f(x) = \frac{\pi^2}{6} - 4 \sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$$