3. Find the Fourier Series for $f(x) = \begin{cases} 1 + \frac{2x}{\pi}, & -\pi < x < 0 \\ 1 - \frac{2x}{\pi}, & 0 < x < \pi \end{cases}$

Consider, $\phi_1(x) = 1 + \frac{2x}{\pi}$, $\phi_2(x) = 1 - \frac{2x}{\pi}$

$\phi_1(-x) = 1 + \frac{2(-x)}{\pi} = 1 - \frac{2x}{\pi} = \phi_2(x)$

$\phi_2(-x) = 1 - \frac{2(-x)}{\pi} = 1 + \frac{2x}{\pi} = \phi_1(x)$

$\therefore f(x)$ is an even function. $\Rightarrow b_n = 0$

$$a_0 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi} \int_{0}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x-\frac{x^2}{\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi-\frac{\pi^2}{\pi}-0\right]$$

$$= \frac{2}{\pi}\left[\pi-\pi\right]=0.$$

$\therefore a_0 = 0.$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

$u = 1-\frac{2x}{\pi}$	$v = \cos nx$
$u' = -\frac{2}{\pi}$	$v_1 = \frac{\sin nx}{n}$
$u'' = 0$	$v_2 = -\frac{\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2}{\pi}\frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} - \frac{2}{\pi}\frac{\cos n\pi}{n^2} - \frac{\sin 0}{n} + \frac{2}{\pi}\frac{\cos 0}{n^2}\right]$$

$$= \frac{2}{\pi}\left[0 - \frac{2}{\pi}\frac{(-1)^n}{n^2} - 0 + \frac{2}{\pi}\frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\,\frac{2}{\pi n^2}\left[1-(-1)^n\right]$$

$$a_n = \frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]$$

$\therefore$ The Fourier series is

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{\pi^2 n^2}\left[1-(-1)^n\right]\cos nx.$$

$$f(x) = \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\left[1-(-1)^n\right]\cos nx$$

4. Find the Fourier Series for $f(x) = \begin{cases} L+x, & -L < x < 0 \\ L-x, & 0 < x < L \end{cases}$

$\phi_1(x) = L+x$ & $\phi_2(x) = L-x$.

$\phi_1(-x) = L-x = \phi_2(x)$ & $\phi_2(-x) = L-(-x) = L+x = \phi_1(x)$

$\therefore$ $f(x)$ is an even function

$\therefore b_n = 0$.

The Fourier Series is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n x \pi}{L}\right)$$

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\,dx$$

$$= \frac{2}{L}\int_{0}^{L} (L-x)\,dx = \frac{2}{L}\left[Lx - \frac{x^2}{2}\right]_0^L$$

$$= \frac{2}{L}\left[L^2 - \frac{L^2}{2}\right] = \frac{2}{L}\frac{L^2}{2}$$

$$a_0 = L$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n x \pi}{L}\right)dx$$

$$= \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \frac{2}{L}\int_{0}^{L} (L-x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$u = L-x$	$v = \cos\left(\frac{n\pi x}{L}\right)$
$u' = -1$	$v_1 = \dfrac{\sin\left(\frac{n\pi x}{L}\right)}{\frac{n\pi}{L}}$
$u'' = 0$	$v_2 = -\dfrac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}$

$$a_n = \frac{2}{L}\left[(L-x)\frac{\sin\left(\frac{n\pi x}{L}\right)}{\left(\frac{n\pi}{L}\right)} - (1)\frac{\cos\left(\frac{n\pi x}{L}\right)}{\frac{n^2\pi^2}{L^2}}\right]_0^L$$

$$= \frac{2}{L}\left[0 - \frac{\cos n\pi}{\frac{n^2\pi^2}{L^2}} - 0 + \frac{\cos 0}{\frac{n^2\pi^2}{L^2}}\right]$$

$$= \frac{2}{L}\left[-\frac{(-1)^n L^2}{n^2\pi^2} + \frac{L^2}{n^2\pi^2}\right]$$

$$x + x^2$$

$$= \frac{2L^2}{L n^2 \pi^2}\left[1 - (-1)^n\right]$$

$$a_0 = \frac{2\pi^2}{3}$$

$$a_n = 4\frac{(-1)^n}{n^2}$$

$$a_n = \frac{2L}{n^2\pi^2}\left[1 - (-1)^n\right]$$

$$b_n = \frac{-2(-1)^n}{n}$$

$$\therefore f(x) = \frac{L}{2} + \sum_{n=1}^{\infty} \frac{2L}{n^2\pi^2}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{L}\right)$$

$$= \frac{L}{2} + \frac{2L}{\pi^2}\sum_{n=1}^{\infty}\left[1-(-1)^n\right]\cos\left(\frac{n\pi x}{L}\right)$$