Odd & Even fns:

Even function:

A real function $f(x)$ is said to be even if $f(x) = f(-x)$.

If $f(x)$ is an even function then

$$\int_{-l}^{l} f(x)\,dx = 2\int_{0}^{l} f(x)\,dx.$$

Odd function:

A real function $f(x)$ is said to be odd if $f(-x) = -f(x)$.

If $f(x)$ is an odd function then

$$\int_{-l}^{l} f(x)\,dx = 0.$$

Note: If $f(x)$ does not satisfies even and odd function then it is called neither even nor odd function

Example:-

1. $f(x) = x^2$

   $f(-x) = (-x)^2 = x^2 = f(x)$

   It is an even function.

2. $f(x) = x \cos x$

   $f(-x) = -x\cos(-x) = -x\cos x = -f(x)$

   It is odd function.

3. $f(x) = x \sin x$ → even function

4. $f(x) = |x|$ → even function

5. $f(x) = x + x^2$ → Neither even nor odd.

Note:-

1. Even fn × Even fn = Even fn
2. Odd fn × Odd fn = Even fn
3. Even fn × odd fn = odd fn
4. For even function, $b_n = 0$
5. For odd function, $a_0 = 0$ & $a_n = 0$

1. Find the Fourier Series for the function $f(x) = |x|$, $-\pi \leq x \leq \pi$

$$f(x) = |x| = x$$

$$f(-x) = |-x| = x = f(x)$$

$\therefore f(x)$ is an even function.

$\therefore b_n = 0.$

The Fourier Series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\,dx \quad [\because f(x) \text{ is even fn}]$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x\,dx = \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \frac{2}{\pi}\frac{\pi^2}{2}$$

$$a_0 = \pi$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \cos nx\,dx$$

$u = x$ | $V = \cos nx$

$u' = 1$ | $V_1 = \frac{\sin nx}{n}$

$u'' = 0$ | $V_2 = -\frac{\cos nx}{n^2}$

$$= \frac{2}{\pi}\left[\frac{x \sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[0 + \frac{\cos n\pi}{n^2} - 0 - \frac{1}{n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$\therefore$ The Fourier Series is $f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[(-1)^n - 1\right] \cos nx$

2. Find the Fourier Series $f(x) = x$ in $(-\pi, \pi)$

$f(x) = x$

$f(-x) = -x = -f(x)$

$\therefore f(x)$ is odd function

$\therefore a_0 = 0$ & $a_n = 0$.

The Fourier series is given by $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx.$$

$$= \frac{2}{\pi}\int_{0}^{\pi} x \sin nx \, dx \quad [\because x \sin nx \text{ is even fn}]$$

$u = x$	$v = \sin nx$
$u' = 1$	$v_1 = -\frac{\cos nx}{n}$
$u'' = 0$	$v_2 = -\frac{\sin nx}{n^2}$

$$b_n = \frac{2}{\pi}\left[-\frac{x \cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$b_n = \frac{2}{\pi}\left[-\frac{\pi \cos n\pi}{n} + 0 + 0 - 0\right]$$

$$b_n = \frac{-2}{n}(-1)^n$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{(-2)(-1)^n}{n} \sin nx$$

5. Find the Fourier Series for $f(x) = |\sin x|$, $-\pi < x < \pi$

Here $\sin x$ is an odd function.

But $f(-x) = |\sin(-x)| = |-\sin x| = \sin x = f(x)$

$\therefore$ $|\sin x|$ is an even function.

$\therefore$ $b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{—①}$$

To find a_0:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x\,dx$$

$$= \frac{2}{\pi}\left[-\cos x\right]_0^{\pi} = \frac{2}{\pi}\left[-\cos\pi + \cos 0\right]$$

$$= \frac{2}{\pi}\left[1+1\right] = \frac{4}{\pi} \quad \text{—②}$$

To find a_n:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx = \frac{2}{\pi}\int_0^{\pi} \sin x \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \cos nx \sin x\,dx$$

W.K.T, $2\cos A \sin B = \sin(A+B) - \sin(A-B)$

$$\therefore a_n = \frac{2}{\pi}\frac{1}{2}\int_0^{\pi}\left[\sin(nx+x) - \sin(nx-x)\right]dx$$

$$= \frac{2}{\pi}\frac{1}{2}\int_0^{\pi}\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\cos n\pi\cos\pi + \sin n\pi \sin\pi}{n+1} + \frac{\cos n\pi \cos\pi + \sin n\pi \sin\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi}{n+1} - \frac{\cos n\pi}{n-1} + \frac{1}{n+1} - \frac{1}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{(n-1)\cos n\pi - (n+1)\cos n\pi + n-1 - n-1}{(n+1)(n-1)}\right]$$

$$= \frac{1}{(n^2-1)\pi}\left[n\cos n\pi - \cos n\pi - n\cos n\pi - \cos n\pi - 2\right]$$

$$= \frac{1}{(n^2-1)\pi}\left[-2\cos n\pi - 2\right] = \frac{-2\left[(-1)^n + 1\right]}{(n^2-1)\pi}$$

$$\therefore a_n = \begin{cases} 0 & , n \text{ in odd} \\ \frac{-4}{\pi(n^2-1)} & , n \text{ is even} \end{cases} \quad \text{— (3)}$$

when $n=1$,

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x \cos x \, dx = \frac{2}{\pi}\frac{1}{2}\int_0^{\pi} \sin 2x \, dx.$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2x}{2}\right]_0^{\pi} = \frac{-1}{2\pi}\left[\cos 2\pi - \cos 0\right]$$

$$= \frac{-1}{2\pi}\left[1-1\right] = 0.$$

$\therefore a_1 = 0.$ — (4)

sub (2), (3) & (4) in (1)

$$f(x) = \frac{a_0}{2} + a_1 \cos x + \sum_{n=2}^{\infty} a_n \cos nx$$

$$f(x) = \frac{\frac{4}{\pi}}{2} + 0 + \sum_{n=2,4}^{\infty} \frac{-4}{\pi(n^2-1)} \cos nx$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2,4}^{\infty} \frac{\cos nx}{(n^2-1)}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\left[\frac{\cos 2x}{3} + \frac{\cos 4x}{15} + \frac{\cos 6x}{35} + \cdots\right]$$