Interval: $[0$ to $2l)$

3. Find the Fourier Series for the fn $f(x) = x^2$ in $[0, 2l)$

$f(x) = x^2$ in $[0, 2l)$

The Fourier Series is given by,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

To find a_0:

$$a_0 = \frac{1}{l}\int_0^{2l} f(x)\,dx = \frac{1}{l}\int_0^{2l} x^2\,dx$$

$$= \frac{1}{l}\left[\frac{x^3}{3}\right]_0^{2l} = \frac{1}{l}\,\frac{8l^3}{3} = \frac{8l^2}{3}$$

To find a_n:

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_0^{2l} x^2 \cos\frac{n\pi x}{l}\,dx$$

$u = x^2$	$v = \cos\frac{n\pi x}{l}$
$u' = 2x$	$v_1 = \frac{\sin\frac{n\pi x}{l}}{n\pi/l}$
$u'' = 2$	$v_2 = -\frac{\cos\frac{n\pi x}{l}}{(n\pi/l)^2}$
$u''' = 0$	$v_3 = -\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)^3}$

$$a_n = \frac{1}{l}\left[\frac{x^2 \sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)} + 2x\left[\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}\right] - 2\,\frac{\sin\frac{n\pi x}{l}}{(n\pi/l)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[0 + 4l\,\frac{1}{\frac{n^2\pi^2}{l^2}} - 0 - 0 - 0 + 0\right]$$

$$= \frac{1}{l}\,\frac{4l^3}{n^2\pi^2} = \frac{4l^2}{n^2\pi^2}$$

To find b_n:

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_0^{2l} x^2 \sin\left(\frac{n\pi x}{l}\right)dx$$

$u = x^2$ | $v = \sin\frac{n\pi x}{l}$

$u' = 2x$ | $v_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 2$ | $v_2 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$u''' = 0$ | $v_3 = \frac{\cos\left(\frac{n\pi x}{l}\right)}{\left(\frac{n\pi}{l}\right)^3}$

$$b_n = \frac{1}{l}\left[\frac{-x^2\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} + 2x\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} + 2\frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{-(2l)^2}{\frac{n\pi}{l}} + 0 + \frac{2}{\frac{n^3\pi^3}{l^3}} - 0 + 0 - \frac{2}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left(\frac{-4l^2 l}{n\pi}\right) = \frac{-4l^2}{n\pi}$$

The Fourier Series is

$$f(x) = \frac{\frac{8l^2}{3}}{2} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi^2}\cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty}\left(\frac{-4l^2}{n\pi}\right)\sin\frac{n\pi x}{l}$$

$$= \frac{4l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2}{n^2\pi}\cos\left(\frac{n\pi x}{l}\right) - \sum_{n=1}^{\infty}\frac{4l^2}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

4. Find the Fourier series for the fn $f(x) = (l-x)^2$ in $[0, 2l)$

The fourier Series is $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$

$$a_0 = \frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$= \frac{1}{l}\int_0^{2l} (l-x)^2\,dx$$

$$= \frac{1}{l}\left[\frac{(l-x)^3}{-3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[\frac{(l-2l)^3}{-3}\right] - \left[\frac{l^3}{-3}\right]$$

$$= \frac{1}{l}\left[\frac{-l^3}{-3} + \frac{l^3}{3}\right]$$

$$= \frac{1}{l}\,\frac{2l^3}{3} = \frac{2l^2}{3}$$

$u = (l-x)^2$ $\quad V = \cos\frac{n\pi x}{l}$

$u' = -2(l-x)$ $\quad V_1 = \frac{\sin\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 2$ $\quad V_2 = \frac{-\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$u''' = 0$ $\quad V_3 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}$

$$a_n = \frac{1}{l}\int_c^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_0^{2l} (l-x)^2\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[(l-x)^2\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)} - \frac{2(l-x)\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} - 2\frac{\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[l^2\frac{\sin n\pi}{\left(\frac{n\pi}{l}\right)} + \frac{2l\cos n\pi 2}{\left(\frac{n\pi}{l}\right)^2} - \frac{2\sin n\pi}{\left(\frac{n\pi}{l}\right)^3}\right] - \left[l^2\frac{\sin 0}{\left(\frac{n\pi}{l}\right)} - \frac{2l\cos n\pi 2}{\left(\frac{n\pi}{l}\right)^2} - \frac{2\sin 0}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left[\frac{2l^3(+1)}{n^2\pi^2} + \frac{2l^3(+1)}{n^2\pi^2}\right] = \frac{4l^2}{n^2\pi^2}$$

$u = (l-x)^2$ $\quad V = \sin\frac{n\pi x}{l}$

$u' = -2(l-x)$ $\quad V_1 = \frac{-\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}}$

$u'' = 2$ $\quad V_2 = \frac{-\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2}$

$u''' = 0$ $\quad V_3 = \frac{\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}$

$$b_n = \frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{1}{l}\int_0^{2l} (l-x)^2\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{-(l-x)^2\cos\frac{n\pi x}{l}}{\frac{n\pi}{l}} - \frac{2(l-x)\sin\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^2} + \frac{2\cos\frac{n\pi x}{l}}{\left(\frac{n\pi}{l}\right)^3}\right]_0^{2l}$$

$$= \frac{1}{l}\left[-l^2\frac{\cos n\pi(2)}{\frac{n\pi}{l}} + 2l\frac{\sin n\pi(2)}{\left(\frac{n\pi}{l}\right)^2} + \frac{2\cos n\pi(2)}{\left(\frac{n\pi}{l}\right)^3}\right] - \left[-l^2\frac{\cos 0}{\frac{n\pi}{l}} - 2l\frac{\sin 0}{\left(\frac{n\pi}{l}\right)^2} + 2\frac{\cos 0}{\left(\frac{n\pi}{l}\right)^3}\right]$$

$$= \frac{1}{l}\left[-\frac{l^3}{n\pi} + 0 + \frac{2l^3}{n^3\pi^3} + \frac{l^3}{n\pi} + 0 - \frac{2l^3}{n^3\pi^3}\right]$$

$$\therefore b_n = 0.$$

$$\therefore f(x) = \frac{\frac{2l^2}{3}}{2} + \sum_{n=1}^{\infty} \frac{4l^2}{n^2\pi^2} \cos\frac{n\pi x}{l} + \sum_{n=1}^{\infty} (0) \sin\frac{n\pi x}{l}$$

$$= \frac{l^2}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos\frac{n\pi x}{l}.$$