## 4.8 Triple Integrals – Volume of Solids

Volume $= \iiint_V$ dzdydx where V is the volume of the given surface.

**Example: 4.56**

**Find the volume of the sphere $x^2 + y^2 + z^2 = a^2$**

**Solution:**

Volume = 8 X volume of the first octant

z varies from 0 to $\sqrt{\mathbf{a^2 - x^2 - y^2}}$

y varies from 0 to $\sqrt{\mathbf{a^2 - x^2}}$

x varies from 0 to $a$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \int_0^{\sqrt{a^2-x^2-y^2}} dzdydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} [Z]_0^{\sqrt{a^2-x^2-y^2}} dydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2 - x^2 - y^2}\, dydx$$

$$= 8\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{\left(\sqrt{a^2 - x^2}\right)^2 - y^2}\, dydx$$

$$= 8\int_0^a \left[\frac{y}{2}\sqrt{a^2 - x^2 - y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$= 8\int_0^a \left(0 + \frac{(a^2-x^2)}{2}\sin^{-1}1 - 0\right) dx$$

$$= 4\int_0^a (a^2 - x^2)\frac{\pi}{2} dx$$

$$= 2\pi\int_0^a (a^2 - x^2)\, dx$$

$$= 2\pi\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= 2\pi \times \frac{2a^3}{3}$$

$$= \frac{4\pi a^3}{3} \text{ cu. units.}$$

**Example: 4.57**

**Find the volume of the ellipsoid** $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

**Solution:**

Volume =8 X volume of the first octant

z varies from 0 to $c\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}$

y varies from 0 to $\sqrt{1 - \frac{x^2}{a^2}}$

x varies from 0 to a

$$V = 8\int_0^a \int_0^{\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dzdydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} [Z]_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dydx$$

$$= 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} c\sqrt{\left(1 - \frac{x^2}{a^2}\right) - \frac{y^2}{b^2}}\, dydx$$

$$= 8c\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{\frac{b^2\left(1-\frac{x^2}{a^2}\right)-y^2}{b^2}}\, dydx$$

$$= \frac{8c}{b}\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \sqrt{b^2\left(1 - \frac{x^2}{a^2}\right) - y^2}\, dydx$$

$$= \frac{8c}{b}\int_0^a \int_0^k \sqrt{k^2 - y^2}\, dydx \qquad \textit{where } k^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$= \frac{8c}{b}\int_0^a \left[\frac{y}{2}\sqrt{k^2 - y^2} + \frac{k^2}{2}\sin^{-1}\frac{y}{k}\right]_0^k dx$$

$$= \frac{8c}{b}\int_0^a \left(0 + \frac{k^2}{2}\sin^{-1}1 - 0\right) dx$$

$$= \frac{8c}{b}\int_0^a \left(\frac{k^2}{2}\right)\frac{\pi}{2}\, dx$$

$$= \frac{2c\pi}{b} \int_0^a k^2 \, dx$$

$$= \frac{2c\pi}{b} \int_0^a b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi \int_0^a \left(1 - \frac{x^2}{a^2}\right) dx$$

$$= 2bc\pi \left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2bc\pi \left[a - \frac{a^3}{3a^2}\right]$$

$$= 2bc\pi \left(a - \frac{a}{3}\right)$$

$$= 2bc\pi \times \frac{2a}{3}$$

$$= \frac{4\pi abc}{3} \, cu.units.$$

**Example: 4.58**

**Find the volume of the tetrahedron bounded by the coordinate planes and $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$**

**Solution:**

Volume = $\iiint_V$ dzdydx

z varies from 0 to $c\left(1 - \frac{x}{a} - \frac{y}{b}\right)$

y varies from 0 to $b\left(1 - \frac{x}{a}\right)$

x varies from 0 to a

$$V = \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} \int_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dzdydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} [Z]_0^{c\left(1-\frac{x}{a}-\frac{y}{b}\right)} dydx$$

$$= \int_0^a \int_0^{b\left(1-\frac{x}{a}\right)} c\left(1-\frac{x}{a}-\frac{y}{b}\right) dydx$$

$$= c\int_0^a \left[y - \frac{xy}{a} - \frac{y^2}{2b}\right]_0^{b\left(1-\frac{x}{a}\right)} dx$$

$$= c\int_0^a b\left[\left(1-\frac{x}{a}\right) - \frac{x}{a}\left(1-\frac{x}{a}\right) - \frac{b^2\left(1-\frac{x}{a}\right)^2}{2b}\right] dx$$

$$= bc\int_0^a \left[\left(1-\frac{x}{a}\right)^2 - \frac{1}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \frac{bc}{2}\int_0^a \left(1-\frac{x}{a}\right)^2 dx$$

$$= \frac{bc}{2}\left[\frac{\left(1-\frac{x}{a}\right)^3}{3\left(-\frac{1}{a}\right)}\right]_0^a$$

$$= -\frac{abc}{6}\left[\left(1-\frac{x}{a}\right)^3\right]_0^a$$

$$= -\frac{abc}{6}(0-1)$$

$$= \frac{abc}{6} \text{ cu. units.}$$

**Example: 4.59**

**Evaluate $\iiint_V dxdydz$ where V is the volume enclosed by the cylinder $x^2 + y^2 = 1$ and the planes $z = 0$ and $z = 2 - x$.**

**Solution:**

In the positive octant, the limits are

z varies from 0 to $2 - x$

x varies from 0 to $\sqrt{1-y^2}$

y varies from $-1$ to $1$

$$\iiint dxdydz = 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}\int_{0}^{2-x} dzdxdy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}[z]_0^{2-x}\, dxdy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{1-y^2}}(2-x)\, dxdy$$

$$= 2\int_{-1}^{1}\left[2x-\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}} dy$$

$$= 2\int_{-1}^{1}\left[2\sqrt{1-y^2}-\left(\frac{1-y^2}{2}\right)\right] dy$$

$$= 4\int_{-1}^{1}\left[\sqrt{1-y^2}\right] dy - \int_{-1}^{1}[1-y^2]\, dy$$

$$= 4\left[\frac{y}{2}\sqrt{1-y^2}+\frac{1}{2}\sin^{-1}y\right]_{-1}^{1} - \left[y-\frac{y^3}{3}\right]_{-1}^{1}$$

$$= 4\left[0+\frac{1}{2}\sin^{-1}1-0-\frac{1}{2}\sin^{-1}(-1)\right]-\left[1-\frac{1}{3}+1-\frac{1}{3}\right]$$

$$= 4\left[\frac{1}{2}\frac{\pi}{2}+\frac{1}{2}\frac{\pi}{2}\right]-\left[2-\frac{2}{3}\right]$$

$$= 4\left(\frac{2\pi}{4}\right)-\frac{4}{3}$$

$$= 2\pi-\frac{4}{3}$$