SECOND ORDER LINEAR DIFFERENTIAL EQUATION HOMOGENEOUS EQUATION OF EULER TYPE LINEAR DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENT.

An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \dots + a_n y = f(x) \quad \text{①}$$

where $a_1, a_2, \dots a_n$ are constants and $f(x)$ is a function of x. eqn ① can be reduced to linear differential equation with constant coefficient by putting the sub. $x = e^z$ (or) $z = \log x$.

$$x\frac{dy}{dx} = D'y \quad \text{where } D' = \frac{d}{dz}.$$

$$x^2 \frac{d^2y}{dx^2} = D'(D'-1)y$$

$$x^3 \frac{d^3y}{dx^3} = D'(D'-1)(D'-2)y.$$

To find C.F complementary function.

1. Roots are Real + different (m_1, m_2) $(m_1 \neq m_2)$ — $Ae^{m_1 x} + Be^{m_2 x}$.

2. Roots are Real and equal $m_1 = m_2 = m$ (say) — $(Ax+B)e^{mx}$ (or) $(A+Bx)e^{mx}$.

3. Roots are imaginary $\alpha \pm i\beta$ — $e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

4. Roots are $\alpha \pm i\beta$ twice (fourth order) — $e^{\alpha x}[(C_1 + C_2 x)\cos\beta x + (C_3 + C_4 x)\sin\beta x]$

Solve $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} = 0$.

Given $(x^2D^2 + xD)y = 0$. where $D = \frac{d}{dx}$

put $x = e^z$ (or) $z = \log x$.

$xD = D'$ $x^2D^2 = D'(D'-1)$

$$[D'(D'-1) + D']y = 0.$$

$$D'^2 y = 0.$$

Auxillary Equation $m^2 = 0$

$m = 0, 0$.

CF : Complimentary Function $= [A + Bz]e^{0z}$

$CF = A + Bz$

$\cancel{y = PI + }$ $\boxed{y = A + B\log x}$

Solve $(x^2D^2 - 3xD + 4)y = x^2\cos(\log x)$

put $x = e^z$ (or) $z = \log x$.

$xD = D'$ $x^2D^2 = D'(D'-1)$

$$[D'(D'-1) - 3D' + 4]y = e^{2z}\cos z.$$

$$[D'^2 - D' - 3D' + 4]y = e^{2z}\cos z$$

$$[D'^2 - 4D' + 4]y = e^{2z}\cos z.$$

A.E : $m^2 - 4m + 4 = 0$.

$(m-2)(m-2) = 0$.

$m = 2, 2.$

$C.F = (A + BZ)e^{2z} = (A + B\log x)x^{2}.$

Particular Integral.

$$P.I = \frac{1}{D'^2 - 4D' + 4} e^{2z}\cos z = e^{2z} \frac{1}{(D'+2)^2 - 4(D'+2) + 4}$$

$$= e^{2z} \frac{1}{D'^2 + 4D' + 4 - 4D' - 8 + 4} \cos z.$$

$$= e^{2z} \frac{1}{D'^2} \cos z = -e^{2z}\cos z.$$

$P.I = -x^2 \cos(\log x)$

$y = C.F + P.I = (A + B\log x)x^2 - x^2\cos(\log x)$

Solve $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} = \frac{12\log x}{x^2}.$

$$x^2\frac{d^2y}{dx^2} + \frac{x^2}{x}\frac{dy}{dx} = 12\log x.$$

$$[x^2D^2 + xD]y = 12\log x.$$

put $x = e^z$ (or) $z = \log x.$

$xD = D'$ $\qquad x^2D^2 = D'(D'-1)$

$$[D'(D'-1) + D']y = 12z.$$

$$[D'^2 - D' + D']y = 12z.$$

$$\underline{D'^2 y = 12z.}$$

A.E: $m^2 = 0 \Rightarrow \boxed{m = 0, 0}$

$$C.F = (A+BZ)e^{0Z} = A+BZ$$

$$C.F = A + B\log x.$$

$$P.I = \frac{1}{D'^2} 12Z = \frac{12}{D'}\left(\frac{Z^2}{2}\right)$$

$$= 6\left(\frac{Z^2}{D'}\right) = 6\left[\frac{Z^3}{3}\right] = 2Z^3 = 2(\log x)^3$$

$$y = A + B\log x + 2(\log x)^3.$$

HIGHER ORDER LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS:

General form of a linear differential equation of the n^{th} order with constant coefficient

$$\frac{d^n y}{dx^n} + K_1 \frac{d^{n-1} y}{dx^{n-1}} + K_2 \frac{d^{n-2} y}{dx^{n-2}} + \dots + K_n y = X.$$

Where $K_1, K_2 \dots K_n$ are constants solution

$y =$ complementary function + particular Integral

$y = C.F + P.I.$

Solve $\frac{d^2 y}{dx^2} - 6\frac{dy}{dx} + 13y = 0.$

$$\frac{d^2}{dx^2} = D^2.$$

Given: $(D^2 - 6D + 13)y = 0$

A.E $m^2 - 6m + 13 = 0$

$b = 6;\ a = 1$

$c = 13$

$$m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{+6 \pm \sqrt{36 - 4(1)(13)}}{2} = \frac{6 \pm \sqrt{36 - 52}}{2}$$

$$= \frac{6}{2} \pm \frac{\sqrt{-16}}{2} = 3 \pm \frac{\sqrt{i^2 4^2}}{2}$$

$$= 3 \pm 2i$$

$$C.F = e^{3x}[A\cos 2x + B\sin 2x]$$

$P.I = 0.$

$$y = e^{3x}[A\cos 2x + B\sin 2x]$$

2) solve $(D^3 - 3D^2 + 3D - 1)y = 0$

Given $(D^3 - 3D^2 + 3D - 1)y = 0$

$m = 1, 1, 1$

$\therefore C.F = (A + Bx + Cx^2)e^x$.

$P.I = 0$

$y = (A + Bx + Cx^2)e^x$.

Solve: $(D^2 - 4D + 13)y = e^{2x}$.

A.E is $m^2 - 4m + 13 = 0$.

$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)}$$

$$= \frac{2 \pm \sqrt{16-52}}{2}$$

$$= \frac{2 \pm (+6)i}{2} = 2 \pm 3i$$

C.F = $2 \pm 3i$

C.F = $e^{+2x}[A\cos 3x + B\sin 3x]$

$$P.I = \frac{1}{D^2 - 4D + 13} e^{2x}$$

[Put $D = 2$]

$$= \frac{1}{4 - 8 + 13} e^{2x} = \frac{1}{9} e^{2x}.$$

$$y = C.F + P.I = e^{2x}[A\cos 3x + B\sin 3x] + \frac{1}{9}e^{2x}$$

Solve $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = x^2 + 3$.

$(D^2 - 5D + 6) = x^2 + 3$.

$m^2 - 5m + 6 = 0$

$m = 2, 3$

C.F = $Ae^{2x} + Be^{3x}$.

$$P.I = \frac{1}{D^2 - 5D + 6}(x^2 + 3)$$

$$= \frac{1}{6\left[1+\frac{D^2-5D}{6}\right]}(x^2+3)$$

$(1+x)^{-1} = 1-x+x^2+\dots$

$$= \frac{1}{6}\left[1+\frac{D^2-5D}{6}\right]^{-1}(x^2+3)$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)+\left(\frac{D^2-5D}{6}\right)^2-\dots\right](x^2+3)$$

$$= \frac{1}{6}\left[1-\left(\frac{D^2-5D}{6}\right)+\left[\frac{D^4}{36}+\frac{25D^2}{36}-\frac{2D^3}{36}\right]+\dots\right](x^2+3)$$

$$= \frac{1}{6}\left[1\,\frac{-D^2+5D}{6}+\frac{25D^2}{36}\right](x^2+3)$$

$$= \frac{1}{6}\left[1+\frac{5D}{6}-\frac{D^2}{6}+\frac{25D^2}{36}\right](x^2+3)$$

$$= \frac{1}{6}\left[1+\frac{5D}{6}+\left[\frac{-6D^2+25D^2}{36}\right]\right](x^2+3)$$

$$= \frac{1}{6}\left[1+\frac{5D}{6}+\frac{19}{36}D^2\right]x^2+3.$$

$$= \frac{1}{6}\left[(x^2+3)+\frac{5}{6}D(x^2+3)+\frac{19}{36}D^2(x^2+3)\right]$$

$$= \frac{1}{6}\left[(x^2+3)+\frac{5}{6}(2x)+\frac{19}{36}(2)\right]$$

$$P.I = \frac{1}{6}\left[x^2+3+\frac{5x}{3}+\frac{19}{18}\right]$$

~~P.I = 1/108 [~~ (or) $y = Ae^{2x}+Be^{2x}+\frac{1}{6}\left[x^2+3+\frac{5x}{3}+\frac{19}{18}\right]$

$$y = Ae^{2x}+Be^{3x}+\frac{1}{108}\left[18x^2+30x+73\right].$$

METHOD OF VARIATION OF PARAMETERS.

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = X \rightarrow (1)$$

$C.F = c_1 f_1 + c_2 f_2$ where c_1, c_2 are constants and f_1, f_2 are functions of x.

Then $P.I = Pf_1 + Qf_2$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_2 f_1'} dx$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'} dx$$

$$y = c_1 f_1 + c_2 f_2 + P.I.$$

Note: The Wronskian of f_1, f_2 of (1)

$$W = \begin{vmatrix} f_1 & f_1' \\ f_2 & f_2' \end{vmatrix} = f_1 f_2' - f_2 f_1'$$

solve $(D^2 + a^2) y = \sec ax$ using Methods of variation of parameters.

Given $(D^2 + a^2) y = \sec ax$

A.E: $m^2 + a^2 = 0$

$m^2 = -a^2$

$m = \pm ai$

$C.F = c_1 \cos ax + c_2 \sin ax.$

$f_1 = \cos ax$ $\quad f_2 = \sin ax$

$f_1' = -a \sin ax$ $\quad f_2' = a \cos ax$

$$f_1 f_2' - f_2 f_1' = a\cos ax \cos ax + \sin ax\, a \sin ax = a$$

$$y = C.F + P.I.$$

$$P.I = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{f_1 f_2' - f_2 f_1'}\,dx = -\int \frac{\sin ax \sec ax}{a}\,dx$$

$$= -\frac{1}{a}\int \tan ax\,dx = -\frac{1}{a}\left[-\frac{\log(\cos ax)}{a}\right]$$

$$P = \frac{1}{a^2}\log\left[\cos ax\right]$$

$$Q = \int \frac{f_1 X}{f_1 f_2' - f_2 f_1'}\,dx = \int \frac{\cos ax \sec ax}{a}\,dx$$

$$= \frac{1}{a}\int dx = \frac{1}{a}x.$$

$$y = C.F + P.I = c_1 \cos ax + c_2 \sin ax + \frac{1}{a^2}\log(\cos ax)\,[\cos ax] + \frac{1}{a}x \sin ax.$$

2) solve $(D^2+a^2)y = \tan ax$ by M.V.P.

$$(D^2+a^2)y = \tan ax$$

$$(m^2+a^2) = 0 \qquad i^2=-1$$

$$m = \pm ai$$

$$C.F = A\cos ax + B\sin ax.$$

$$f_1 = \cos ax \qquad f_2 = \sin ax$$

$$f_1' = -a\sin ax \qquad f_2' = a\cos ax$$

$$f_1f_2' - f_2f_1' = a\cos^2 ax + a\sin^2 ax = a.$$

$$P.I = Pf_1 + Qf_2$$

$$P = -\int \frac{f_2 X}{f_1f_2' - f_2f_1'}dx = -\int \frac{\sin ax(\tan ax)}{a}dx$$

$$= -\frac{1}{a}\int \frac{\sin^2 ax}{\cos ax}dx = -\frac{1}{a}\int \frac{1-\cos^2 ax}{\cos ax}dx.$$

$\therefore [\sin^2 ax = 1-\cos^2 ax]$

$$= -\frac{1}{a}\int \sec ax + \frac{1}{a}\int \cos ax\, dx$$

$$= -\frac{1}{a}\cdot\frac{1}{a}\log(\sec ax + \tan ax) + \frac{1}{a}\cdot\frac{1}{a}\sin ax$$

$$P = -\frac{1}{a^2}\log[\sec ax + \tan ax] + \frac{1}{a^2}\sin ax.$$

(diff $\cos = -\sin$
integ $\cos = \sin$)

$$Q = \int \frac{f_1 X}{f_1f_2' - f_2f_1'}dx = \int \frac{\cos ax \tan ax}{a}dx$$

$\int \sec ax = [\log(\sec ax + \tan ax)]$

$$= \frac{1}{a}\int \sin ax \, dx = -\frac{1}{a^2}\cos ax .$$

$$P.I = \cos ax \left[\frac{1}{a^2} \sin ax - \frac{1}{a^2} \log(\sec ax + \tan ax) \right]$$

$$\cancel{+\sec ax} - \frac{\cos ax}{a^2} .$$

$$y = C.F + P.I .$$

$$= A\cos ax + B\sin ax + \cos ax$$

$$\left[\frac{1}{a^2} \sin ax - \frac{1}{a^2} \log(\sec ax + \tan ax) - \frac{\cos ax}{a^2} \right]$$

Legendre's Linear Differential Equation.

put $ax+b=e^z$ (or) $z=\log(ax+b)$

$(ax+b)D = aD'$ and $(ax+b)^2D^2 = a^2D'(D'-1)$

in $(ax+b)^n \frac{d^ny}{dx^n} + K_1(ax+b)^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \dots + K_ny = 0.$

① solve $(3x+2)^2\frac{d^2y}{dx^2} + 3(3x+2)\frac{dy}{dx} - 36y = 3x^2+4x+1$

put $3x+2=e^z$ (or) $z=\log(3x+2)$

$x = \frac{1}{3}(e^z-2)$

$(3x+2)D = 3D'$ and $(3x+2)^2D^2 = 9D'(D'-1)$

$[9D'(D'-1)+3(3D')-36]y = 3\left[\frac{1}{3}e^z - \frac{2}{3}\right]^2$

$[9D'^2 - 9D' + 9D' - 36]y = 3\left[\frac{e^{2z}}{9} + \frac{4}{9} - \frac{4}{9}e^z\right] + 4\left(\frac{1}{3}(e^z-2)\right) + 1$

$[9D'^2 - 36]y = \frac{1}{3}e^{2z} - \frac{4}{3}e^z + \frac{4}{3} + \frac{4}{3}e^z - \frac{8}{3} + 1.$

$9[D'^2-4]y = \frac{1}{3}[e^{2z}] - \frac{4}{3} + 1$

$= \frac{1}{3}e^{2z} - \frac{1}{3}.$

$[D'^2-4]y = \frac{1}{27}e^{2z} - \frac{1}{27}. \Rightarrow Ax^2+Bx.$

$C.F = A(3x+2)^2 + B(3x+2)^{-2}$

$P.I_1 = \frac{\log(3x+2)}{108}(3x+2)^2$

$P.I_2 = -\frac{1}{108}$; $y = C.F + P.I_1 + P.I_2.$