SNS COLLEGE OF ENGINEERING

Kurumbapalayam (Po), Coimbatore – 641 107

AN AUTONOMOUS INSTITUTION

Accredited by NBA – AICTE and Accredited by NAAC – UGC with ‘A’ Grade

Approved by AICTE, New Delhi & Affiliated to Anna University, Chennai

TOPIC: 4.1 – PARTIAL DERIVATIVE AND TOTAL DERIVATIVES

Problems based on Partial derivatives

1. If $u = (x-y)(y-z)(z-x)$, then show that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$

Given $u = (x-y)(y-z)(z-x)$

$$\frac{\partial u}{\partial x} = (y-z)\left[(x-y)(-1) + (z-x)(1)\right]$$

$$= -(x-y)(y-z) + (y-z)(z-x)$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$\frac{\partial u}{\partial y} = (z-x)\left[(x-y)(1) + (y-z)(-1)\right]$$

$$= (x-y)(z-x) - (y-z)(z-x)$$

$$\frac{\partial u}{\partial z} = (x-y)\left[(y-z)(+1) + (z-x)(-1)\right]$$

$$= (x-y)(y-z) - (x-y)(z-x)$$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$$

Euler's Theorem for homogeneous function

If u is a homogeneous function of degree 'n' in x and y, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

(1) If $u = \sin^{-1}\left[\frac{x^3-y^3}{x+y}\right]$, then prove that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\tan u$.

Let $f(x,y) = \sin u = \frac{x^3-y^3}{x+y}$

$$f(tx,ty) = \frac{t^3x^3 - t^3y^3}{tx+ty} = t^2 f(x,y)$$

$\therefore$ f is a homogeneous function of degree 2 in x & y.

$\therefore$ By Euler's theorem, we get

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf = 2f \rightarrow (1)$$

Here $f = \sin u$

sub. in ①.

$$x \frac{\partial}{\partial x}(\sin u) + y \frac{\partial}{\partial y}(\sin u) = 2 \sin u$$

$$x \left[\cos u \frac{\partial u}{\partial x}\right] + y \left[\cos u \frac{\partial u}{\partial y}\right] = 2 \sin u.$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 2 \tan u.$$

② If $u = \frac{x}{y} + \frac{y}{x} + \frac{z}{x}$, then find $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z}$.

Given $u(x, y, z) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}$

$$u(tx, ty, tz) = \frac{tx}{ty} + \frac{ty}{tz} + \frac{tz}{tx}$$

$$= t^{0} u(x, y, z)$$

$\therefore$ u is a homogeneous function of x, y, z in degree 0.

$\therefore$ By Euler's theorem,

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = nu$$

$$= 0.$$

4. If $u = \cos^{-1}\left[\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right]$, then prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\frac{1}{2}\cot u.$$

Let $f(x,y) = \cos u = \dfrac{x+y}{\sqrt{x}+\sqrt{y}}$

$$f(tx,ty) = \frac{tx+ty}{\sqrt{tx}+\sqrt{ty}} = t^{1/2} f(x,y)$$

$\Rightarrow$ f is a homogeneous function of degree $\frac{1}{2}$ in x and y.

∴ By Euler's theorem.

$$x \frac{\partial f}{\partial x} + y \frac{\partial f}{\partial y} = nf = \frac{1}{2} f \rightarrow (1)$$

Here $f = \cos u$, sub in (1).

$$x \frac{\partial}{\partial x}(\cos u) + y \frac{\partial}{\partial y}(\cos u) = \frac{1}{2} \cos u$$

$$x \left[-\sin u \frac{\partial u}{\partial x} \right] + y \left[-\sin u \frac{\partial u}{\partial y} \right] = \frac{1}{2} \cos u$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = -\frac{1}{2} \frac{\cos u}{\sin u}$$

$$= -\frac{1}{2} \cot u$$

7. If $u = (x^2+y^2+z^2)^{-1/2}$, then find the value of $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$.

Given $u = (x^2+y^2+z^2)^{-1/2}$

$$\frac{\partial u}{\partial x} = -\frac{1}{2}(x^2+y^2+z^2)^{-3/2}(2x)$$

$$\frac{\partial u}{\partial x} = -x(x^2+y^2+z^2)^{-3/2}$$

$$\frac{\partial^2 u}{\partial x^2} = -\left[x\left(-\frac{3}{\cancel{2}}\right)(x^2+y^2+z^2)^{-5/2}(\cancel{2}x) + (x^2+y^2+z^2)^{-3/2}\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 3x^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \rightarrow (1)$$

lly,

$$\frac{\partial^2 u}{\partial y^2} = 3y^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \rightarrow (2)$$

$$\frac{\partial^2 u}{\partial z^2} = 3z^2(x^2+y^2+z^2)^{-5/2} - (x^2+y^2+z^2)^{-3/2} \rightarrow (3)$$

(1) + (2) + (3)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= (x^2+y^2+z^2)^{-5/2}\left[3x^2+3y^2+3z^2\right] - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)(x^2+y^2+z^2)^{-5/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 3(x^2+y^2+z^2)^{-3/2} - 3(x^2+y^2+z^2)^{-3/2}$$

$$= 0$$

Total Derivatives

If $u = f(x,y)$, where $x = \phi(t)$ and $y = \psi(t)$ then we can express u as a function of t alone by substituting the values of x and y in $f(x,y)$. Thus, we can find the ordinary derivative $\frac{du}{dt}$ which is called the total derivative of u to distinguish it from the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$. i.e) $\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt}$

Composite function of one variable

If $u = f(x, y, z)$ where x, y, z are all functions of a variable t, then

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

Differentiation of Implicit functions

If $f(x, y) = c$ be an implicit relation between x and y which defines as a differentiable function of x, then

$$\frac{dy}{dx} = \frac{-\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} \qquad \left[\because \frac{\partial f}{\partial y} \neq 0\right]$$

Compositive function of two variables

If $z = f(x, y)$ where $x = \phi(u, v)$, $y = \psi(u, v)$, then z is a function of u, v

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

(1) Find $\frac{dy}{dx}$ when $x^3 + y^3 = 3axy$.

Let $f(x, y) = x^3 + y^3 - 3axy$

$$\frac{dy}{dx} = -\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial y}} = -\frac{3x^2 - 3ay}{3y^2 - 3ax}$$

$$= -\frac{x^2 - ay}{y^2 - ax}$$

(5) If $Z = f(y-z, z-x, x-y)$, show that $\frac{\partial Z}{\partial x} + \frac{\partial Z}{\partial y} + \frac{\partial Z}{\partial z} = 0$.

Let $u = y - z$, $v = z - x$, $w = x - y$

$$Z = f(u, v, w)$$

$$\frac{\partial Z}{\partial x} = \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{\partial f}{\partial u}(0) + \frac{\partial f}{\partial v}(-1) + \frac{\partial f}{\partial w}(1)$$

$$= -\frac{\partial f}{\partial v} + \frac{\partial f}{\partial w}$$

IIIly $\frac{\partial Z}{\partial y} = +\frac{\partial f}{\partial u} - \frac{\partial f}{\partial w}$

(11) If $u = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, prove that $u_{xx} + u_{yy} = 0$.

$$u_x = \frac{\partial u}{\partial x} = \frac{1}{x^2+y^2}(2x) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left[-\frac{y}{x^2}\right]$$

$$= \frac{2x}{x^2+y^2} + \frac{1}{\frac{x^2+y^2}{x^2}}\left[-\frac{y}{x^2}\right]$$

$$= \frac{2x}{x^2+y^2} - \frac{y}{x^2+y^2} = \frac{2x-y}{x^2+y^2}$$

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{(x^2+y^2)(2) - (2x-y)(2x)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2-4x^2+2xy}{(x^2+y^2)^2}$$

$$u_{xx} = \frac{2y^2 - 2x^2 + 2xy}{(x^2+y^2)^2} \longrightarrow (1)$$

$$u_y = \frac{\partial u}{\partial y} = \frac{1}{x^2+y^2}(2y) + \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{1}{\frac{x^2+y^2}{x^2}}\left(\frac{1}{x}\right)$$

$$= \frac{2y}{x^2+y^2} + \frac{x}{x^2+y^2} = \frac{2y+x}{x^2+y^2}$$

$$u_{yy} = \frac{(x^2+y^2)(2) - (2y+x)(2y)}{(x^2+y^2)^2}$$

$$= \frac{2x^2+2y^2 - 4y^2 - 2xy}{(x^2+y^2)^2}$$

$$u_{yy} = \frac{2x^2 - 2y^2 - 2xy}{(x^2+y^2)^2} \longrightarrow (2)$$

(1) + (2)

$$u_{xx} + u_{yy} = \frac{2y^2 - 2x^2 + 2xy + 2x^2 - 2y^2 - 2xy}{(x^2+y^2)^2}$$

$$= 0$$