Topic:4.8–LAGRANGE'S METHOD

LAGRANGE'S METHOD OF UNDETERMINED MULTIPLIERS.

Def: Suppose, We require to find the maximum and minimum values of $f(x,y,z)$ where x, y, z are subject to a constraint equation $g(x,y,z)=0$.

We define a function

$$F(x,y,z,\lambda) = f(x,y,z) + \lambda g(x,y,z) \rightarrow (1)$$

Where λ is called Lagrange multiplier which is independent of x, y, z.

The necessary conditions for a maximum or Minimum are

$$\frac{\partial F}{\partial x} = 0 \rightarrow (2)$$

$$\frac{\partial F}{\partial y} = 0 \rightarrow (3)$$

$$\frac{\partial F}{\partial z} = 0 \rightarrow (4).$$

Solving the four equations for four unknowns λ, x, y, z, we obtain the point (x, y, z). The point may be a maxima, minima or neither which is decided by physical consideration. This method is applicable when we have more than one constraint eqn connecting the variables.

Problems based on Lagrange's method of undetermined Multipliers.

1) Find the minimum value of $x^2+y^2+z^2$ subject to the condition $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

Solution: Let the auxiliary function 'F' be

$$F(x,y,z,\lambda) = (x^2+y^2+z^2) + \lambda\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-1\right)$$

Where λ is lagrange multipliers.

$$F_x = \frac{\partial F}{\partial x} \quad ; \quad F_y = \frac{\partial F}{\partial y} \quad ; \quad F_z = \frac{\partial F}{\partial z}.$$

$$= 2x + \lambda\left(\frac{-1}{x^2}\right) \qquad = 2y + \lambda\left(\frac{-1}{y^2}\right) \qquad = 2z + \lambda\left(\frac{-1}{z^2}\right)$$

$$= 2x - \frac{\lambda}{x^2} \qquad = 2y - \frac{\lambda}{y^2} \qquad = 2z - \frac{\lambda}{z^2}.$$

For a minimum at (x, y, z) we have.

$F_x = 0$; $F_y = 0$; $F_z = 0$

$$2x - \frac{\lambda}{x^2} = 0 \qquad 2y - \frac{\lambda}{y^2} = 0 \qquad 2z - \frac{\lambda}{z^2} = 0.$$

$$2x = \frac{\lambda}{x^2} \qquad 2y = \frac{\lambda}{y^2} \qquad 2z = \frac{\lambda}{z^2}$$

$$x^3 = \frac{\lambda}{2} \qquad y^3 = \frac{\lambda}{2} \qquad z^3 = \frac{\lambda}{2}$$

$$x = \left(\frac{\lambda}{2}\right)^{1/3} \qquad y = \left(\frac{\lambda}{2}\right)^{1/3} \qquad z = \left(\frac{\lambda}{2}\right)^{1/3}$$

From (1), (2), (3), we get. $x = y = z$.

Given: $\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$.

$$3\left(\frac{1}{x}\right) = 1. \qquad [\text{Since } x = y = z].$$

$$3 = x$$

$$\Rightarrow y = 3 \text{ \& } z = 3.$$

$\therefore (3, 3, 3)$ is the point. Where minimum value occur. The minimum value is $3^2 + 3^2 + 3^2 = 27$.