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Topic: 3.8 – ENVELOPES

Envelope:

A curve which touches each member of a family of curve is called the envelope of that family of curves,

The envelope of a family of curves is the locus of the ultimate points of intersection of the consecutive members of the family.

Method 1: for finding envelope:

1) If the family of curves is expressed as a quadratic equation of the parameter, say,
 $A\lambda^2 + B\lambda + C = 0$ where A, B, C are functions of x and y and λ is the parameter then the envelope of this family is given by $B^2 - 4AC = 0$.

2) Analytic method to find the Envelope of the family of curves.

1. Differentiate $f(x, y, c) = 0$ partially w.r.t the parameters c .

2. Eliminate 'c' from $f(x, y, c) = 0$ + $\frac{\partial f(x, y, c)}{\partial c} = 0$

We get the envelope of the family.



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Problems:

1. Find the envelope of the family of straight lines
 $y = mx + am^2$; m being the parameter,

Soln: Given $y = mx + am^2$,
 $am^2 + mx - y = 0$.

This is quadratic in m , so the envelope

is $B^2 - 4AC = 0$, here $A = a$
 $B = x$
 $C = -y$

$$x^2 - 4a(-y) = 0$$

$$\Rightarrow x^2 + 4ay = 0$$

2. Find the envelope of the family of lines
 $y = mx + \frac{a}{m}$ where 'a' is a constant.

Soln: Given $y = mx + \frac{a}{m}$
 $y = \frac{m^2x + a}{m}$

$$my = m^2x + a$$

$$m^2x - my + a = 0$$

This is a quadratic in 'm'.

So the envelope is $B^2 - 4AC = 0$

$$(-y)^2 - 4xa = 0$$

$$(i.e) y^2 = 4ax$$



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3. Find the envelope of the family of straight lines. $x \cos \theta + y \sin \theta = a$, where θ being parameter

soln:

Given $x \cos \theta + y \sin \theta = a \rightarrow \textcircled{1}$

$\textcircled{1}$ diff. w.r.t $\Rightarrow -x \sin \theta + y \cos \theta = 0 \rightarrow \textcircled{2}$

Squaring and adding $\textcircled{1} + \textcircled{2}$ the envelope

$$x^2 (\cos^2 \theta + \sin^2 \theta) + y^2 (\sin^2 \theta + \cos^2 \theta) = a^2$$

(i.e) $x^2 + y^2 = a^2$, which is a circle.

4. Find the envelope of the family of straight lines $y = mx + \sqrt{a^2 m^2 + b^2}$ where 'm' is the parameter.

soln:

$$y - mx = \sqrt{a^2 m^2 + b^2}$$

$$(y - mx)^2 = a^2 m^2 + b^2 \Rightarrow y^2 - 2mxy + m^2 x^2 = a^2 m^2 + b^2$$

$$m^2 (x^2 - a^2) - 2mxy + y^2 - b^2 = 0$$

which is quadratic in 'm',

here $A = x^2 - a^2$; $B = -2xy$; $C = y^2 - b^2$

$$B^2 - 4AC = 4x^2 y^2 - 4(x^2 - a^2)(y^2 - b^2) = 0$$

$$4x^2 y^2 - 4(x^2 y^2 - x^2 b^2 - a^2 y^2 + a^2 b^2) = 0$$

$$4x^2 y^2 - 4x^2 y^2 + 4x^2 b^2 + 4a^2 y^2 - 4a^2 b^2 = 0$$

$$4x^2 b^2 + 4a^2 y^2 = 4a^2 b^2$$

$$\therefore \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



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5. Find the envelope of the family of lines
 $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$; θ being the parameter.

Soln: Given $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1 \rightarrow (1)$

Diff. p.w.r. to (1) w.r.t. ' θ ' we get.

$$-\frac{x}{a} \sin \theta + \frac{y}{b} \cos \theta = 0 \rightarrow (2)$$

Squaring and adding (1) + (2)

$$\left(\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta\right)^2 + \left(-\frac{x}{a} \sin \theta + \frac{y}{b} \cos \theta\right)^2 = 1^2 + 0^2$$

$$\frac{x^2}{a^2} \cos^2 \theta + \frac{y^2}{b^2} \sin^2 \theta + \frac{2xy}{ab} \cos \theta \sin \theta$$

$$+ \frac{x^2}{a^2} \sin^2 \theta + \frac{y^2}{b^2} \cos^2 \theta - \frac{2xy}{ab} \cos \theta \sin \theta = 1$$

$$\frac{x^2}{a^2} [\cos^2 \theta + \sin^2 \theta] + \frac{y^2}{b^2} [\cos^2 \theta + \sin^2 \theta] = 1$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



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b. Find the envelope of $x \sec \theta - y \tan \theta = a$ where ' θ ' being the parameter.

Soln:

$$\text{Given } x \sec \theta - y \tan \theta = a \rightarrow (1)$$

$$x \sec \theta = a + y \tan \theta.$$

Squaring both sides.

$$x^2 \sec^2 \theta = a^2 + 2ay \tan \theta + y^2 \tan^2 \theta.$$

$$x^2 (1 + \tan^2 \theta) = a^2 + 2ay \tan \theta + y^2 \tan^2 \theta.$$

$$x^2 + x^2 \tan^2 \theta = y^2 \tan^2 \theta + 2ay \tan \theta + a^2$$

$$(y^2 - x^2) \tan^2 \theta + 2ay \tan \theta + (a^2 - x^2) = 0$$

$$(i.e.) (y^2 - x^2) m^2 + 2aym + (a^2 - x^2) = 0.$$

Where $m = \tan \theta$ which is a quadratic form in m .

$$\text{Here } A = y^2 - x^2, B = 2ay; C = a^2 - x^2$$

The envelope is $B^2 - 4AC = 0$.

$$4a^2 y^2 - 4(y^2 - x^2)(a^2 - x^2) = 0$$

$$4a^2 y^2 - 4[a^2 y^2 - x^2 y^2 - x^2 a^2 + x^4] = 0$$

$$4a^2 y^2 - 4a^2 y^2 + 4x^2 y^2 + 4x^2 a^2 - 4x^4 = 0$$

$$\therefore \text{by } x^2 \Rightarrow y^2 + a^2 - x^2 = 0$$

$$x^2 - y^2 = a^2.$$