Topic: 1.3 – PROBLEMS ON EIGEN VALUES AND EIGEN VECTORS

Find the Eigen values and Eigen Vectors of $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$.

Soln:

Let $A = \begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$

Step 1: To find the char. eqn.

The char. eqn. of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

where S_1 = Sum of the main diagonal elements

$= -2+1+0$

$= -1$

S_2 = Sum of the minors of the main diagonal elements.

$= \begin{vmatrix} 1 & -6 \\ -2 & 0 \end{vmatrix} + \begin{vmatrix} -2 & -3 \\ -1 & 0 \end{vmatrix} + \begin{vmatrix} -2 & 2 \\ 2 & 1 \end{vmatrix}$

$= (0-12) + (0-3) + (-2-4)$

$= -12-3-6 = -21$

$S_3 = |A|$

$= -2(0-12) - 2(0-6) - 3(-4+1)$

$= 24+12+9 = 45$

$\therefore$ The char. eqn. is $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

Step 2: To solve the char. equ.

$$\lambda^3+\lambda^2-21\lambda-45=0 \quad \text{—— (1)}$$

If $\lambda=1$ then (1) $\Rightarrow 1+1-21-45 \neq 0$

If $\lambda=-1$ then (1) $\Rightarrow -1+1+21-45 \neq 0$

If $\lambda=2$, (1) $\Rightarrow 8+4-42-45 \neq 0$

If $\lambda=-2$, (1) $\Rightarrow -8+4+42-45 \neq 0$

If $\lambda=3$, (1) $\Rightarrow 27+9-63-45 \neq 0$

If $\lambda=-3$, (1) $\Rightarrow -27+9+63-45 = 0$

Hence $\lambda=-3$ is a root of $\lambda^3+\lambda^2-21\lambda-45=0$.

$$\begin{array}{r|rrrr} -3 & 1 & 1 & -21 & -45 \\ & 0 & -3 & 6 & 45 \\ \hline & 1 & -2 & -15 & \boxed{0} \end{array}$$

$$\lambda^3+\lambda^2-21\lambda-45 = (\lambda+3)(\lambda^2-2\lambda-15)=0$$

(ie) $(\lambda+3)(\lambda+3)(\lambda-5)=0$

$$\lambda=-3,\,-3,\,5$$

Step: 3 To find the Eigenvectors.

Solve $(A-\lambda I)x=0$

$$\begin{bmatrix} -2-\lambda & 2 & -3 \\ 2 & 1-\lambda & -6 \\ -1 & -2 & 0-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{—— (A)}$$

Case(i):

If $\lambda = -3$ then equ. (A) becomes.

$$\begin{pmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + 2x_2 - 3x_3 = 0$ ——— (2)

$2x_1 + 4x_2 - 6x_3 = 0$ ——— (3)

$-x_1 - 2x_2 + 3x_3 = 0$ ——— (4)

Here (1), (2) & (3) are same equ.

We consider $x_1 + 2x_2 - 3x_3 = 0$

put $x_1 = 0$ we get $2x_2 = 3x_3$

$$\frac{x_2}{3} = \frac{x_3}{2}$$

$\therefore$ Eigen vector is $X_1 = \begin{pmatrix} 0 \\ 3 \\ 2 \end{pmatrix}$

Put $x_2 = 0$, we get $x_1 - 3x_3 = 0$

$x_1 = 3x_3 \Rightarrow \frac{x_1}{3} = \frac{x_3}{1}$

$\therefore$ the Eigenvector $X_2 = \begin{pmatrix} 3 \\ 0 \\ 1 \end{pmatrix}$

Case: 2:

If $\lambda = 5$ then equ. (A) becomes.

If $\lambda = 5$ then equ. (A) becomes.

$$\begin{pmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$-7x_1 + 2x_2 - 3x_3 = 0 \quad \text{——— (5)}$$

$$2x_1 - 4x_2 - 6x_3 = 0 \quad \text{——— (6)}$$

$$-x_1 - 2x_2 - 5x_3 = 0 \quad \text{——— (7)}$$

Solving (5) & (6) we get

$$\begin{array}{cccccc} & x_1 & x_2 & x_3 & & \\ -7 & 2 & -3 & -7 & 2 & \\ 2 & -4 & -6 & 2 & -4 & \end{array}$$

$$\frac{x_1}{-12-12} = \frac{x_2}{-6-42} = \frac{x_3}{28-4}$$

$$\frac{x_1}{-24} = \frac{x_2}{-48} = \frac{x_3}{24}$$

$$\frac{x_1}{1} = \frac{x_2}{2} = \frac{x_3}{-1}$$

$\therefore$ Eigen vector $X_3 = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$

Find the Eigenvalues and Eigenvectors of

$$\begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$$

Solu: Let $A = \begin{pmatrix} 6 & -6 & 5 \\ 14 & -13 & 10 \\ 7 & -6 & 4 \end{pmatrix}$

Step1: To find the Char. eqn.

The Char. eqn. of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$

$$S_1 = 6 - 13 + 4 = -3$$

$$S_2 = \begin{vmatrix} -13 & 10 \\ -6 & 4 \end{vmatrix} + \begin{vmatrix} 6 & 5 \\ 7 & 4 \end{vmatrix} + \begin{vmatrix} 6 & -6 \\ 14 & -13 \end{vmatrix}$$

$$= (-52 + 60) + (24 - 35) + (-78 + 84)$$

$$= 8 - 11 + 6 = 3$$

$$S_3 = |A| = -1$$

$\therefore$ The Char eqn. is $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = 0$ — (1)

Step 2: To find Eigen values.

If $\lambda = 1$, $(1) \Rightarrow -1+3+3+1 \neq 0$.

If $\lambda = -1$, $(1) \Rightarrow -1+3-3+1 = 0$.

$\therefore (\lambda+1)$ is a factor

$$\begin{array}{r|rrrr} -1 & 1 & 3 & 3 & 1 \\ & 0 & -1 & -2 & -1 \\ \hline & 1 & 2 & 1 & \boxed{0} \end{array}$$

$$(\lambda+1)(\lambda^2+2\lambda+1) = 0$$

$$(\lambda+1)(\lambda+1)(\lambda+1) = 0$$

Hence the Eigenvalues are $-1, -1, -1$

Step: 3 To find the Eigenvector,

Solve $(A-\lambda I)X = 0$

$$\begin{pmatrix} 6-\lambda & -6 & 5 \\ 14 & -13-\lambda & 10 \\ 7 & -6 & 4-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad \text{--- (A)}$$

When $\lambda = -1$, (A) becomes.

$$\begin{pmatrix} 7 & -6 & 5 \\ 14 & -12 & 10 \\ 7 & -6 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$7x_1 - 6x_2 + 5x_3 = 0$ ——— (2)

$14x_1 - 12x_2 + 10x_3 = 0$ ——— (3)

$7x_1 - 6x_2 + 5x_3 = 0$ ——— (4)

the above 3 equs. are same

put $x_1 = 0$ in (2) we get $-6x_2 = 5x_3$

$\frac{x_2}{5} = \frac{x_3}{6}$

Hence, Eigenvector $X_1 = \begin{pmatrix} 0 \\ 5 \\ 6 \end{pmatrix}$

put $x_2 = 0$ in (2), we get $7x_1 + 5x_3 = 0$

$7x_1 = -5x_3$

$\frac{x_1}{-5} = \frac{x_3}{7}$

Hence, Eigenvector $X_2 = \begin{pmatrix} -5 \\ 0 \\ 7 \end{pmatrix}$

put $x_3 = 0$ in (2), we get $7x_1 - 6x_2 = 0$ (ie) $7x_1 = 6x_2$

$\frac{x_1}{6} = \frac{x_2}{7}$

Hence, Eigenvector $X_3 = \begin{pmatrix} 6 \\ 7 \\ 0 \end{pmatrix}$

Find the Eigenvalues and Eigenvectors of $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Soln: Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$

Step 1: To find the characteristic equ.

The Char: equ. of A is $\lambda^3 - S_1\lambda^2 + S_2\lambda - S_3 = 0$.

where $S_1 = 0+0+0 = 0$

$$S_2 = \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix} + \begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}$$

$$= (0-1) + (0-1) + (0-1)$$

$$= -3$$

$$S_3 = |A| = 0(0-1) - 1(0-1) + 1(1-0)$$

$$= 0+1+1 = 2$$

$\therefore$ The Char. equ. is $\lambda^3 - 0\lambda^2 - 3\lambda - 2 = 0$

$$\lambda^3 - 3\lambda - 2 = 0$$

Step 2: To find Eigen values.

Solving $\lambda^3 - 3\lambda - 2 = 0$ ——— (1)

If $\lambda = 1$, (1) $\Rightarrow 1 - 3 - 2 \neq 0$.

If $\lambda = -1$, (1) $\Rightarrow -1 + 3 - 2 = 0$

$\therefore \lambda = -1$ is a root

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & 0 & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

(ie) $(\lambda+1)(\lambda^2 - \lambda - 2) = 0$

$(\lambda+1)(\lambda+1)(\lambda-2)=0$

Hence the Eigenvalues are $-1, -1, 2$.

Step: 3 To find Eigenvector.

Solve $(A-\lambda I)X = 0$

(ie) $\begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ — (A)

Case (i) $\lambda = 2$. (A) becomes

$$\begin{bmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$-2x_1 + x_2 + x_3 = 0$ ——— (2)

$x_1 - 2x_2 + x_3 = 0$ ——— (3)

$x_1 + x_2 - 2x_3 = 0$ ——— (4)

Solving (2) & (3)

$$\frac{x_1}{1+2} = \frac{x_2}{1+2} = \frac{x_3}{4-1}$$

$$\frac{x_1}{3} = \frac{x_2}{3} = \frac{x_3}{3}$$

(ie) $\frac{x_1}{1} = \frac{x_2}{1} = \frac{x_3}{1}$

$\therefore$ Eigenvector $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Case (ii). If $\lambda = -1$ then eqn. (A) becomes.

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$x_1 + x_2 + x_3 = 0$ ——— (5)

$x_1 + x_2 + x_3 = 0$ ——— (6)

$x_1 + x_2 + x_3 = 0$ —— (7)

Here (5), (6) & (7) are same equ.

Put $x_1 = 0$ we get $x_2 = -x_3$

$$\frac{x_2}{1} = \frac{x_3}{-1}$$

$\therefore$ Eigen vector $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$ as X_3 is orthogonal to X_1 and X_2

Since the given matrix is Symmetric.

$[1 \ 1 \ 1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$ (or) $l + m + n = 0$ —— (8)

$[0 \ 1 \ -1] \begin{bmatrix} l \\ m \\ n \end{bmatrix} = 0$ (or) $0l + m - n = 0$ —— (9)

Solving (8) & (9)

$$\frac{l}{-1-1} = \frac{m}{0+1} = \frac{n}{1-0}$$

$$\frac{l}{-2} = \frac{m}{1} = \frac{n}{1}$$

(ie) $l/2 = \frac{m}{-1} = \frac{n}{-1}$

Hence, Eigenvector $X_3 = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$

Result:

1. Eigenvalues of A are $(2, -1, -1)$
2. Eigen vector are $X_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $X_2 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ & $X_3 = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}$