Topic: 1.9 – NATURE OF QUADRATIC FORM

Reduction of quadratic form to Canonical form by orthogonal transformation - Nature of quadratic form.

Quadratic form:

A homogeneous polynomial of the second degree in any number of variables is called a quadratic form.

Eg: $2x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 + 5x_1x_3 - 6x_2x_3$ is a quadratic form of three variable.

Note: The matrix corresponding to the quadratic form is

$$\begin{bmatrix} \text{coeff } x_1^2 & \frac{1}{2}\text{coeff. } x_1x_2 & \frac{1}{2}\text{coeff. } x_1x_3 \\ \frac{1}{2}\text{coeff. } x_2x_1 & \text{coeff. } x_2^2 & \frac{1}{2}\text{coeff. } x_2x_3 \\ \frac{1}{2}\text{coeff. } x_3x_1 & \frac{1}{2}\text{coeff. } x_3x_2 & \text{coeff. } x_3^2 \end{bmatrix}$$

Problem:

Write the matrix of the quadratic form

$2x_1^2 - 2x_2^2 + 4x_3^2 + 2x_1x_2 - 6x_1x_3 + 6x_2x_3$

Soln:

$$Q = \begin{bmatrix} \text{coeff. } x_1^2 & \frac{1}{2}\text{coeff. } x_1x_2 & \frac{1}{2}\text{coeff. } x_1x_3 \\ \frac{1}{2}\text{coeff. } x_2x_1 & \text{coeff. } x_2^2 & \frac{1}{2}\text{coeff. } x_2x_3 \\ \frac{1}{2}\text{coeff. } x_3x_1 & \frac{1}{2}\text{coeff. } x_3x_2 & \text{coeff. } x_3^2 \end{bmatrix}$$

Here $x_2x_1 = x_1x_2$
$x_3x_1 = x_1x_3$
$x_2x_3 = x_3x_2$

$$\therefore Q = \begin{bmatrix} 2 & 1 & -3 \\ 1 & -2 & 3 \\ -3 & 3 & 4 \end{bmatrix}.$$

Write the matrix of the quadratic form
$2x^2 + 8z^2 + 4xy + 10xz - 2yz$

Soln:

$$Q = \begin{bmatrix} \text{coeff. } x^2 & \frac{1}{2}\text{coeff. } x.y & \frac{1}{2}\text{coeff. } xz \\ \frac{1}{2}\text{coeff. } yx & \text{coeff. } y^2 & \frac{1}{2}\text{coeff. } yz \\ \frac{1}{2}\text{coeff. } zx & \frac{1}{2}\text{coeff. } zy & \text{coeff. } z^2 \end{bmatrix}$$

$$Q = \begin{bmatrix} 2 & 2 & 5 \\ 2 & 0 & -1 \\ 5 & -1 & 8 \end{bmatrix}$$

Write the quadratic form corresponding to the following symmetric matrix $\begin{bmatrix} 0 & -1 & 2 \\ -1 & 1 & 4 \\ 2 & 4 & 3 \end{bmatrix}$

Soln.

$\therefore$ General form is

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2(a_{12})x_1x_2 + 2(a_{23})x_2x_3 + 2a_{13}x_1x_3$$

$$= 8x_1^2 + x_2^2 + 3x_3^2 - 2x_1x_2 + 4x_1x_3 + 8x_2x_3.$$

Quadratic form as a product of Matrices.

Let $A = [a_{ij}]$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $X^T = [x_1, x_2 \dots x_n]$

If we have a quadratic form $Q = \sum_{j=1}^{n} \sum_{i=1}^{n} a_{ji} x_i x_j$, where $a_{ij} = a_{ji}$ then Q can be expressed as

$Q = X^T A X$ where the symmetric matrix

$A = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ - & - & - & - \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ is called the matrix of the Q.F.

Canonical form of a Quadratic form.

Let $Q = X^T A X$ be a quadratic form in n variables $x_1, x_2 \dots x_n$.

Let $X = NY$ be a orthogonal linear transformation, where N is a normalized matrix

Now $Q = X^T A X = (NY)^T A (NY)$

$= Y^T (N^T A N) Y$,

$$= Y^T D Y$$

$$= (y_1, y_2 \ldots y_n) \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots \lambda_n y_n^2$$

(ie) $\therefore$ which is the canonical form of the quadratic form

Not: This form is also called diagonalization of the quadratic form (or) to express the quadratic form as 'sum of squares'.

Nature of Quadratic form:

Rank of A:

When the quadratic form is reduced to the canonical form it contains only r terms which is the rank of A

Index of the Q.F (s):

The number of positive square terms in the canonical form is called the index of the quadratic form.

Signature of the Q.F

The difference of number of positive and negative square terms is called the signature of the quadratic form [= s-(r-s) = 2s-r]

Note:

The Quadratic form $Q = x^T A x$ in n variables is said to be

(i) positive definite, if $r = n$ and $s = n$ (or) if all the eigenvalues of A are positive.

(ii) negative definite, if $r = n$ & $s = 0$ (or) if all the eigenvalues of A are $-ve$.

(iii) positive semidefinite, if $r < n$ and $s = r$ (or) if all the Eigenvalues of $A \geq 0$ & atleast one Eigenvalue is zero.

(iv) Negative semidefinite if $r < n$ & $s = 0$ (or) if all the eigenvalues of $A \leq 0$ & at least one Eigenvalue is zero.

(v) Indefinite, in all other cases (or) if A has both $+ve$ & $-ve$ Eigenvalues.

Test for Nature of a Quadratic form through principal minors.

Let $A = [a_{ij}]$ be the matrix of the quadratic form in n variables $x_1, x_2 \ldots x_n$. Then A is a square symmetric matrix of order n.

Let $D_1 = |a_{11}|$

$$D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$\cdots\cdots$

$D_n = |A|$

Here $D_1, D_2 \ldots D_n$ are the principal minors of A.

(i) The Q.F is positive definite if $D_1, D_2 \ldots D_n$ are all +ve i.e) $D_n > 0 \ \forall n$.

(ii) The Q.F is –ve definite if $D_1, D_3 \ldots$ are all –ve and $D_2, D_4, D_6 \ldots$ are all +ve

i.e) $(-1)^n D_n > 0 \ \forall n$.

(iii) The Q.F is +ve semi-definite if $D_n \geq 0$ & atleast one $D_i = 0$

(iv) The Q.F is –ve semidefinite if $(-1)^n D_n \geq 0$ & atleast one $D_i = 0$.

(v) The Q.F is indefinite in all other cases.