Topic: 1.7 – REDUCTION TO QUADRATIC FORM TO CANONICAL FORM

Reduction of Quadratic form to Canonical form.

Working Rule:

1. Write the matrix of the given Q.F
2. To find the Cha. Equ.
3. To solve the Cha. Equ.
4. To find the Eigenvectors orthogonal to each other.
5. Form Normalised matrix N.
6. Find N^T.
7. Find AN
8. Find $D = N^T AN$
9. Canonical form $[y_1\ y_2\ y_3]\,[D]\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.

Reduce the quadratic form $Q = 6x^2+3y^2+3z^2-4xy-2yz+4zx$ into canonical form by an orthogonal transformation. Also discuss its nature.

Soln:

Given Q.F: $6x^2+3y^2+3z^2-4xy-2yz+4zx$

Step 1. The matrix of the Q.F is

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$

Step 2. To find the Cha. Equation.

The Cha. equ. of A is $\lambda^3 - s_1\lambda^2 + s_2\lambda - s_3 = 0$.

$= (9-1) + (18-4) + (18-4)$
$= 8 + 14 + 14 = 36$

$S_3 \cdot |A| = 6(9-1) + 2(-6+2) + 2(2-6)$
$= 6(8) + 2(-4) + 2(-4)$
$= 48 - 8 - 8 = 32$

$\therefore$ The cha. eqn is $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$

Step 3: To solve the cha. eqn

$\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 0$ — (1)

If $\lambda = 1$, then $\lambda^3 - 12\lambda^2 + 36\lambda - 32 = 1 - 12 + 36 - 32 \neq 0$

If $\lambda = 1$, (1) $\Rightarrow -1 - 12 - 36 - 32 \neq 0$

If $\lambda = 2$, then (1) $\Rightarrow 8 - 48 + 72 - 32 = 0$

$\therefore \lambda = 2$ is a root

By synthetic division

$$\begin{array}{c|cccc} 2 & 1 & -12 & 36 & -32 \\ & 0 & 2 & -20 & 32 \\ \hline & 1 & -10 & 16 & \underline{0} \end{array}$$

$\therefore$ (1) $\Rightarrow (\lambda - 2)(\lambda^2 - 10\lambda + 16) = 0$

$(\lambda - 2)(\lambda - 2)(\lambda - 8) = 0$

$\lambda = 2, 2, 8$

Step 4: To find the Eigenvector:

Solve $(A-\lambda I)X=0$.

$$\begin{bmatrix} 6-\lambda & -2 & 2 \\ -2 & 3-\lambda & -1 \\ 2 & -1 & 3-\lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{---} \; (A)$$

Case (i) If $\lambda = 8$ then (A) becomes

$$\begin{bmatrix} -2 & -2 & 2 \\ -2 & -5 & -1 \\ 2 & -1 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x_1 - 2x_2 + 2x_3 = 0 \quad \text{---} \; (2)$$

$$-2x_1 - 5x_2 - x_3 = 0 \quad \text{---} \; (3)$$

$$2x_1 - x_2 - 5x_3 = 0 \quad \text{---} \; (4)$$

Solving (2) & (3)

$$\begin{array}{cccccc} & & x_1 & x_2 & x_3 & \\ -2 & -2 & 2 & -2 & -2 & \\ -2 & -5 & -1 & -2 & -5 & \end{array}$$

$$\frac{x_1}{2+10} = \frac{x_2}{-4-2} = \frac{x_3}{10-4}$$

$$\frac{x_1}{12} = \frac{x_2}{-6} = \frac{x_3}{6}$$

i.e. $\dfrac{x_1}{2} = \dfrac{x_2}{-1} = \dfrac{x_3}{1} \Rightarrow X_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$

Case (ii) when $\lambda = 2$, (A) becomes.

$$\begin{bmatrix} 4 & -2 & 2 \\ -2 & 1 & -1 \\ 2 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$4x_1 - 2x_2 + 2x_3 = 0$ —— (5)

$-2x_1 + x_2 - x_3 = 0$ —— (6)

$2x_1 - x_2 + x_3 = 0$ —— (7)

(5), (6) & (7) are same as

$2x_1 - x_2 + x_3 = 0$

If $x_1 = 0$ we get $-x_2 + x_3 = 0$

$-x_2 = -x_3$

$\frac{x_2}{1} = \frac{x_3}{1} \Rightarrow X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

The third Eigenvector orthogonal to X_1 & X_2 since the matrix A is symmetric

Let $X_3 = \begin{bmatrix} l \\ m \\ n \end{bmatrix}$.

X_3 is orthogonal to X_1 & X_2

$\Rightarrow X_1^T X_3 = 0 \Rightarrow 2l - m + n = 0$ — (8)

& $X_2^T X_3 = 0 \Rightarrow 0l + m + n = 0$ — (9)

Solving (8) & (9)

$$\begin{array}{cccccc} & & l & m & n & \\ 2 & -1 & 1 & 2 & -1 & \\ 0 & 1 & 1 & 0 & 1 & \end{array}$$

$$\frac{l}{-1-1} = \frac{m}{0-2} = \frac{n}{2-0}$$

$$\frac{l}{-2} = \frac{m}{-2} = \frac{n}{2}$$

(ie) $\frac{l}{1} = \frac{m}{1} = \frac{n}{-1} \implies X_3 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

Step: 5 Form Normalised matrix

$$N = \begin{bmatrix} \frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{3}} \end{bmatrix}$$

Step: 6 : Find N^T

$$N^T = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix}$$

Step: 7 : Find AN

$$AN = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{3} \\ -1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}$$

$$= \begin{pmatrix} \frac{12+2+2}{\sqrt{6}} & \frac{0-2+2}{\sqrt{2}} & \frac{6-2-2}{\sqrt{3}} \\ \frac{-4-3-1}{\sqrt{6}} & \frac{0+3-1}{\sqrt{2}} & \frac{-2+3+1}{\sqrt{3}} \\ \frac{4+1+3}{\sqrt{6}} & \frac{0-1+3}{\sqrt{2}} & \frac{2-1-3}{\sqrt{3}} \end{pmatrix} = \begin{pmatrix} 16/\sqrt{6} & 0 & 2/\sqrt{3} \\ -8/\sqrt{6} & 2/\sqrt{2} & 2/\sqrt{3} \\ 8/\sqrt{6} & 2/\sqrt{2} & -2/\sqrt{3} \end{pmatrix}$$

Step: 8 Find $N^T A N$

$$D = N^T A N = \begin{bmatrix} 2/\sqrt{6} & -1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{3} & 1/\sqrt{3} & -1/\sqrt{3} \end{bmatrix} \begin{bmatrix} 16/\sqrt{6} & 0 & 2/\sqrt{3} \\ -8/\sqrt{6} & 2/\sqrt{2} & 2/\sqrt{3} \\ 8/\sqrt{6} & 2/\sqrt{2} & -2/\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{32+8+8}{6} & \frac{0-12+2}{\sqrt{12}} & \frac{4-2-2}{\sqrt{18}} \\ \frac{0-8+8}{\sqrt{12}} & \frac{0+2+2}{2} & \frac{0+2-2}{\sqrt{6}} \\ \frac{16-8-8}{\sqrt{18}} & \frac{0+2-2}{\sqrt{6}} & \frac{2+2+2}{3} \end{bmatrix}$$

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Step: 9 Canonical form: $(y_1\ y_2\ y_3)\ (D)\ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}$

$$(y_1\ y_2\ y_3) \begin{pmatrix} 8 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = 8y_1^2 + 2y_2^2 + 2y_3^2$$

Step: 10 Nature of the Q.F

Since all the Eigen values of Given matrix A are +ve, $\therefore$ Q.F is +ve definite