TOPIC : 2 - Tautology & Logical Equivalence

Tautology

A statement which is true always irrespective of the truth values of the individual variables is called a tautology.

Example $P \lor \neg P$ is a Tautology.

Contradiction

A statement which is always false is called a contradiction.

Example $P \land \neg P$ is a contradiction.

Contigency

A statement which is neither Tautology nor contradiction is called Contigency.

(1) Show that $Q \lor (P \land \neg Q) \lor (\neg P \land \neg Q)$ is a tautology.

P	Q	$\neg P$	$\neg Q$	$P \land \neg Q$	$Q \lor (P \land \neg Q)$	$\neg P \land \neg Q$	S
T	T	F	F	F	T	F	T
T	F	F	T	T	T	F	T
F	T	T	F	F	T	F	T
F	F	T	T	F	F	T	T

$\therefore$ Given statement is Tautology.

③ Show that $(P \to Q) \wedge (Q \to R) \to (P \to R)$ is a tautology.

P	Q	R	$P \to Q$	$Q \to R$	$(P\to Q)\wedge(Q\to R)$	$P \to R$	S
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence the given statement is a tautology.

④ Prove $((p \vee q) \wedge \neg(\neg p \wedge (\neg q \vee \neg r))) \vee (\neg p \wedge \neg q) \vee (\neg p \wedge \neg r)$ is a tautology.

p	q	r	$\neg p$ (1)	$\neg q$ (2)	$\neg r$ (3)	$p \vee q$ (4)	$\neg q \vee \neg r$ (5)	$\neg p \wedge \neg q \vee \neg r$ (6)	$\neg(\neg p \wedge \neg q \vee \neg r)$ (7)	(4) ∧ (7) (8)	(1) ∧ (2) (9)	(8) ∨ (9)	(1) ∧ (3) (10)	(9) ∨ (10)
T	T	T	F	F	F	T	F	F	T	T	F	T	F	T
T	T	F	F	F	T	T	T	F	T	T	F	T	F	T
T	F	T	F	T	F	T	T	F	T	T	F	T	F	T
T	F	F	F	T	T	T	T	F	T	T	F	T	F	T
F	T	T	T	F	F	T	F	F	T	T	F	T	F	T
F	T	F	T	F	T	T	T	~~F~~T	F	F	F	F	T	T
F	F	T	T	T	F	F	T	T	F	F	T	T	F	T
F	F	F	T	T	T	F	T	T	F	F	T	T	T	T

Equivalence

Two statements P and Q are equivalent iff $P \leftrightarrow Q$ or $P \rightleftarrows Q$ is a tautology. It is denoted by the symbol $P \Leftrightarrow Q$ which is read as "P is equivalent to Q".

Logical Equivalences (or) Equivalence rules	
Idempotent Laws	$p \wedge p \Leftrightarrow p \qquad p \vee p \Leftrightarrow p$
Associative Laws	$(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$ $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
Commutative Laws	$p \wedge q \Leftrightarrow q \wedge p$ $p \vee q \Leftrightarrow q \vee p$
De Morgan's Laws	$\neg(p \wedge q) \Leftrightarrow (\neg p \vee \neg q)$ $\neg(p \vee q) \Leftrightarrow (\neg p \wedge \neg q)$
Distributive Laws	$p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r)$ $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r)$
Complement Laws	$p \wedge \neg p \Leftrightarrow F$ $p \vee \neg p \Leftrightarrow T$
Absorption Laws	$p \vee (p \wedge q) \Leftrightarrow p$ $p \wedge (p \vee q) \Leftrightarrow p$
contrapositive Law	$p \rightarrow q \Leftrightarrow \neg q \rightarrow \neg p$
conditional as disjunction	$p \rightarrow q \Leftrightarrow \neg p \vee q$
Biconditional as conditional	$p \leftrightarrow q \Leftrightarrow (p \rightarrow q) \wedge (q \rightarrow p)$

① Show that $(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R) \Leftrightarrow$

$(\neg P \wedge (\neg Q \wedge R)) \vee (Q \wedge R) \vee (P \wedge R)$

$\Leftrightarrow (\neg P \wedge (\neg Q \wedge R)) \vee ((Q \vee P) \wedge R)$ ($\because$ Distributive law)

$\Leftrightarrow ((\neg P \wedge \neg Q) \wedge R) \vee ((Q \vee P) \wedge R)$ ($\because$ Associative law)

$\Leftrightarrow [(\neg P \wedge \neg Q) \vee (Q \vee P)] \wedge R$ ($\because$ Distributive law)

$\Leftrightarrow [\neg(P \vee Q) \vee (P \vee Q)] \wedge R$ ($\because$ De morgan's law)

$\Leftrightarrow T \wedge R$ $[\because P \vee \neg P \Leftrightarrow T]$

$\Leftrightarrow R$ $[\because P \wedge T \Leftrightarrow P]$

(8) Show that $(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q)) \Leftrightarrow (\neg P \ldots$

$(P \vee Q) \wedge (\neg P \wedge (\neg P \wedge Q))$

$\Leftrightarrow (P \vee Q) \wedge ((\neg P \wedge \neg P) \wedge Q)$ [∵ Associative laws]

$\Leftrightarrow (P \vee Q) \wedge (\neg P) \wedge Q)$ [∵ Idemponent law]

$\Leftrightarrow (P \vee (\neg P \wedge Q)) \wedge (Q \vee (\neg P \wedge Q))$

[∵ Distributive law]

$\Leftrightarrow ((P \wedge \neg P) \wedge Q) \wedge (Q \vee (\neg P \wedge Q))$

[∵ Associative law]

$\Leftrightarrow (T \wedge Q) \wedge (Q \vee (Q \wedge \neg P))$ [∵ commutative]

$\Leftrightarrow Q \wedge ((Q \vee Q) \wedge \neg P)$ [∵ Associative]

$\Leftrightarrow Q \wedge (Q \wedge \neg P)$ [∵ Idempotent]

$\Leftrightarrow (Q \wedge Q) \wedge \neg P$ [∵ Associative]

$\Leftrightarrow Q \wedge \neg P$ [∵ Idempotent]

Tautological Implication

A statement P is said to be tautologically imply a statement Q iff $P \rightarrow Q$ is a tautology. we shall denote this idea by $P \Rightarrow$

(2) Prove that $(P \rightarrow Q) \wedge (R \rightarrow Q) \Rightarrow (P \vee R) \rightarrow Q$

T.S.T $(P \rightarrow Q) \wedge (R \rightarrow Q) \rightarrow ((P \vee R) \rightarrow Q)$ is a tautology.

$(P \rightarrow Q) \wedge (R \rightarrow Q) \rightarrow ((P \vee R) \rightarrow Q)$

$\Leftrightarrow (\neg P \vee Q) \wedge (\neg R \vee Q) \rightarrow (\neg (P \vee R) \vee Q)$ [∵ $P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow ((\neg P \wedge \neg R) \vee Q) \rightarrow (\neg (P \vee R) \vee Q)$ [∵ Distributive]

$\Leftrightarrow (\neg (P \vee R) \vee Q) \rightarrow (\neg (P \vee R) \vee Q)$ [Demorgan's law]

$\Leftrightarrow \neg(\neg (P \vee R) \vee Q) \vee (\neg (P \vee R) \vee Q)$ [∵ $P \rightarrow Q \Leftrightarrow \neg P \vee Q$]

$\Leftrightarrow T$ [$\neg P \vee P \Leftrightarrow T$]

③ Show that $((P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))) \vee (\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$ is a tautology.

Consider $(P \vee Q) \wedge \neg(\neg P \wedge (\neg Q \vee \neg R))$

$\Leftrightarrow (P \vee Q) \wedge \neg(\neg P \wedge \neg(Q \wedge R))$ [∵ Demorgan's Law]

$\Leftrightarrow (P \vee Q) \wedge \neg(\neg(P \vee (Q \wedge R)))$ [∵ Demorgan's law]

$\Leftrightarrow (P \vee Q) \wedge (P \vee (Q \wedge R))$ [∵ Double negation]

$\Leftrightarrow \cancel{(P \vee Q)} \cancel{\wedge} (P \vee Q) \wedge (P \vee R)$ [∵ Distributive law]

$\Leftrightarrow P \vee (Q \wedge (Q \wedge R))$

$\Leftrightarrow P \vee ((Q \wedge Q) \wedge R)$ [∵ Associative law]

$\Leftrightarrow P \vee (Q \wedge R)$ [∵ Idempotent law]

Now

$(\neg P \wedge \neg Q) \vee (\neg P \wedge \neg R)$

$\Leftrightarrow \neg(P \vee Q) \vee \neg(P \vee R)$ [Demorgan's law]

$\Leftrightarrow \neg((P \vee Q) \wedge (P \vee R))$ [Demorgan's law]

$\Leftrightarrow \neg(P \vee (Q \wedge R))$ [∵ Distributive law]

Now $(P \vee (Q \wedge R)) \vee \neg(P \vee (Q \wedge R)) \Leftrightarrow T$

Hence the gn. equation is a tautology.