TOPIC : 2 - FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Formation of PDE by Elimination of arbitrary constants Functions:

$$\phi(u, v) = 0 \Rightarrow \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0.$$

① Eliminate the arbitrary function f from $z = f\left(\frac{y}{x}\right)$ form a PDE.

Sol:

Given $z = f\left(\frac{y}{x}\right)$

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2}$$

$$\Rightarrow p = f'\left(\frac{y}{x}\right) \cdot \frac{-y}{x^2} \rightarrow ①$$

Diff p.w.r.to y, we get

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x}$$

$$\Rightarrow q = f'\left(\frac{y}{x}\right) \cdot \frac{1}{x} \rightarrow ②$$

from ① & ②.

$$\frac{p}{q} = \frac{-y}{x^2} \cdot \frac{x}{1}$$

$$\frac{p}{q} = \frac{-y}{x}$$

$$px = -qy$$

$$\Rightarrow px + qy = 0.$$

(2) $z = f(xy)$

Sol: $z = f(xy)$

Diff w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'(xy) \cdot y$$

$$p = f'(xy) \cdot y$$

Diff w.r.to y, we get

$$\frac{\partial z}{\partial y} = f'(xy) \cdot x$$

$$q = f'(xy) \cdot x$$

$$\frac{p}{q} = \frac{y}{x}$$

$$px = qy$$

$$\Rightarrow px - qy = 0.$$

(3) $\phi\left(z^2 - xy, \frac{x}{z}\right) = 0$.

Sol:

Here $u = z^2 - xy$; $v = \frac{x}{z}$

$$\frac{\partial u}{\partial x} = 2z\frac{\partial z}{\partial x} - y = 2zp - y$$

$$\frac{\partial u}{\partial y} = 2z\frac{\partial z}{\partial y} - x = 2zq - x$$

$$\frac{\partial v}{\partial x} = \frac{z \cdot 1 - x \cdot \frac{\partial z}{\partial x}}{z^2} = \frac{z - xp}{z^2}$$

$$\frac{\partial v}{\partial y} = \frac{z \cdot 0 - x \cdot q}{z^2} = \frac{-xq}{z^2}$$

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 2zp-y & \frac{z-xp}{z^2} \\ 2zq-x & -\frac{xq}{z^2} \end{vmatrix} = 0$$

$$(2zp-y)\left(-\frac{xq}{z^2}\right) - (2zq-x)\left(\frac{z-xp}{z^2}\right) = 0$$

$$\frac{1}{z^2}\left[-2z\cancel{p}xq + yxq - 2zqz + 2zq\cancel{x}p + xz - x^2p\right] = 0$$

$$xyq + xz - 2z^2q - x^2p = 0$$

$$xz = 2z^2q + x^2p - xyq$$

$$xz = x^2p \mp q(xy - 2z^2)$$

4. $z = f(x^2+y^2)$.

Sol:

$$z = f(x^2+y^2)$$

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = f'(x^2+y^2)\cdot 2x$$

$$p = 2x\, f'(x^2+y^2)$$

Diff p.w.r.to y, we get

$$\frac{\partial z}{\partial y} = f'(x^2+y^2)\cdot 2y$$

$$q = 2y\, f'(x^2+y^2)$$

$$\frac{p}{q} = \frac{\cancel{2}x\, \cancel{f'(x^2+y^2)}}{\cancel{2}y\, \cancel{f'(x^2+y^2)}}$$

$$\frac{p}{q} = \frac{x}{y}$$

$$py - qx = 0.$$

5. $z = xf(2x+y) + g(2x+y)$

Sol:

$$z = xf(2x+y) + g(2x+y)$$

Diff p. w. r. to x.

$$p = \frac{\partial z}{\partial x} = f(2x+y) + xf'(2x+y)\cdot 2 + g'(2x+y)\cdot 2 \rightarrow (1)$$

$$q = xf'(2x+y) + g'(2x+y) \rightarrow (2)$$

$$r = f'(2x+y)\cdot 2 + xf''(2x+y)\cdot 4 + f'(2x+y)\cdot 2$$
$$+ g''(2x+y)\cdot 4 \rightarrow (3)$$

$$t = xf''(2x+y) + g''(2x+y) \rightarrow (4)$$

$$s = f'(2x+y) + 2xf''(2x+y) + 2g''(2x+y) \rightarrow (5)$$

(3) $\Rightarrow$ $r = 4f'(2x+y) + 4[xf''(2x+y) + g''(2x+y)]$

$$r = 4f'(2x+y) + 4t \rightarrow (6)$$

(5) $\Rightarrow$ $s = 2[xf''(2x+y) + g''(2x+y)] + f'(2x+y)$

$$s = 2t + f'(2x+y) \rightarrow (7)$$

(6) $-$ 4 $\times$ (7)

$$r = 4f'(2x+y) + 4t$$
$$4s = 4f'(2x+y) + 8t$$
$$(-) \quad (-) \quad (-)$$

$$r - 4s = -4t$$

$$r - 4s + 4t = 0.$$

(6) From the p.de by eliminating ϕ from

$\phi(x^2+y^2+z^2, ax+by+cz)=0$.

Sol: Here $u = x^2+y^2+z^2$ $\quad v = ax+by+cz$.

$\frac{\partial u}{\partial x} = 2x + 2zp$ $\quad \frac{\partial v}{\partial x} = a + cp$

$\frac{\partial u}{\partial y} = 2y + 2zq$ $\quad \frac{\partial v}{\partial y} = b + cq$.

$$\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} \end{vmatrix} = 0$$

$$\begin{vmatrix} 2x+2zp & a+cp \\ 2y+2zq & b+cq \end{vmatrix} = 0$$

$2(x+zp)(b+cq) - 2(y+zq)(a+cp) = 0$.