TOPIC : 10 – COMPLEX FORM

Formula: Complex form $(-\pi, \pi)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where $c_n = \frac{1}{b-a} \int_a^b f(x) e^{-inx} \, dx.$

Formula: complex form $(-l, l)$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

where $c_n = \frac{1}{\cancel{b-a}\ 2l} \int_{-l}^{l} f(x) e^{\frac{-in\pi x}{l}} \, dx.$

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① Find the complex form $f(x) = e^{ax}, -\pi < x < \pi$ in the form $e^{ax} = \frac{\sinh a\pi}{\pi} \sum_{n=-\infty}^{\infty} (-1)^n \frac{a+in}{a^2+n^2} e^{inx}$ & hence prove that $\frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$.

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

$$c_n = \frac{1}{b-a} \int_a^b f(x) e^{-inx} dx.$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ax} e^{-inx} dx$$

$$= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{(a-in)x} dx$$

$$c_n = \frac{1}{2\pi}\left[\frac{e^{(a-in)\pi}}{a-in} - \frac{e^{(a-in)(-\pi)}}{a-in}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}\cdot e^{-in\pi} - e^{-a\pi}\cdot e^{in\pi}\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(\cos n\pi - i\sin n\pi) - e^{-a\pi}(\cos n\pi + i\sin n\pi)\right]$$

$$= \frac{1}{2\pi(a-in)}\left[e^{a\pi}(-1)^n - e^{-a\pi}(-1)^n\right]$$

$$= \frac{(-1)^n}{2\pi(a-in)}\left(e^{a\pi} - e^{-a\pi}\right)$$

$$= \frac{(-1)^n}{\pi(a-in)}\,\frac{\sinh a\pi}{a+in}\cdot a+in$$

$$c_n = \frac{(-1)^n(a+in)\sinh a\pi}{\pi(a^2+n^2)}$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)\sinh a\pi}{\pi(a^2+n^2)}\,e^{inx}$$

$$e^{ax} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n(a+in)\sinh a\pi}{\pi(a^2+n^2)}\,e^{inx}.$$

Put $x=0$.

$$1 = \sum^{\infty} \frac{(-1)^n(a+in)\sinh a\pi}{}$$

Equating the real parts, we get

$$\frac{\pi}{\sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n a}{a^2+n^2}$$

$$\therefore \frac{\pi}{a \sinh a\pi} = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{a^2+n^2}$$

2. $f(x) = e^{ax}$ in $(-l, l)$

<u>Sol:</u> $$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}}$$

$$c_n = \frac{1}{2l}\int_{-l}^{l} f(x) e^{\frac{-inx\pi}{l}} dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{ax} e^{\frac{-in\pi x}{l}} dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(a-\frac{in\pi}{l}\right)x} dx$$

$$= \frac{1}{2l}\left[\frac{e^{\left(a-\frac{in\pi}{l}\right)x}}{a-\frac{in\pi}{l}}\right]_{-l}^{l}$$

$$= \frac{1}{2l}\left[\frac{e^{\frac{al-in\pi}{l}\cdot l}}{a-\frac{in\pi}{l}} - \frac{e^{\frac{al-in\pi}{l}\cdot(-l)}}{a-\frac{in\pi}{l}}\right]$$

$$= \frac{1}{\quad} \left[e^{al-in\pi} \quad e^{-al \quad in\pi} \right]$$

$$C_n = \frac{1}{2l\frac{(al-in\pi)}{l}}\left[e^{al}\cdot(-1)^n - e^{-al}(-1)^n\right]$$

$$= \frac{(-1)^n}{2(al-in\pi)}\left[e^{al} - e^{-al}\right]$$

$$C_n = \frac{(-1)^n}{(al-in\pi)} \sinh al$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n \sinh al}{(al-in\pi)} e^{+\frac{in\pi x}{l}}$$

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(3) Find the complex form of the fourier series of $f(x) = e^{-x}$ in $-1 < x < 1$.

Sol:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{l}} \qquad 2l = 2,\ l = 1.$$

$$C_n = \frac{1}{b-a}\int_a^b f(x)\, e^{-\frac{in\pi x}{l}}\, dx$$

$$c_n = \frac{1}{2}\int_{-1}^{1} e^{-x}\cdot e^{-in\pi x}\, dx$$

$$= \frac{1}{2}\int_{-1}^{1} e^{-(1+in\pi)x}\, dx$$

$$= \frac{1}{2}\left[\frac{e^{-(1+in\pi)x}}{\ldots}\right]_{-1}^{1}$$

$$c_n = \frac{1}{2(1+in\pi)}\left[e^{+(1+in\pi)} - e^{-(1+in\pi)}\right]$$

$$= \frac{1}{2(1+in\pi)}\left[e^{1}(\cos n\pi + i\sin n\pi) - e^{-1}(\cos n\pi - i\sin n\pi)\right]$$

$$= \frac{1}{2(1+in\pi)}\left[(-1)^n e - e^{-1}(-1)^n\right]$$

$$= \frac{(-1)^n}{2(1+in\pi)}\left[e - e^{-1}\right]$$

$$= \frac{(-1)^n}{1+in\pi}\left(\frac{e-e^{-1}}{2}\right)$$

$$c_n = \frac{(-1)^n}{1+in\pi} \sinh 1$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+in\pi} \sinh 1 \, e^{in\pi x} \cdot \frac{1-in\pi}{1-in\pi}$$

$$f(x) = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{1+n^2\pi^2} \sinh 1 \, e^{in\pi x}(1-in\pi)$$