



TOPIC : 9 – PARSEVAL'S THEOREM

Parseval's Theorem:
Let $f(x)$ be a periodic function with period 2π , defined in the interval $(-\pi, \pi)$. Then

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} [f(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2} \sum_{n=1}^{\infty} [a_n^2 + b_n^2]$$

Root mean Square value:
The root mean square value of $f(x)$ over the interval (a, b) is defined as

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}}$$

Problems:

① $f(x) = x^2$ in $(0, \pi)$.

Sol:

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}} = \sqrt{\frac{\int_0^{\pi} (x^2)^2 dx}{\pi}}$$
$$= \sqrt{\frac{(x^5) \frac{\pi}{5}}{\pi}} = \sqrt{\frac{\pi^5}{5\pi}} = \sqrt{\frac{\pi^4}{5}}$$
$$R.M.S = \frac{\pi^2}{\sqrt{5}}$$

② $f(x) = 1-x$ in $0 < x < 1$.

Sol:

$$R.M.S = \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}} = \sqrt{\frac{\int_0^1 (1-x)^2 dx}{1}}$$



② $f(x) = \pi - x$ in $0 < x < 2\pi$

Sol:

$$\begin{aligned} \text{R.M.S} &= \sqrt{\frac{\int_a^b [f(x)]^2 dx}{b-a}} = \sqrt{\frac{\int_0^{2\pi} (\pi-x)^2 dx}{2\pi}} \\ &= \sqrt{\frac{\left(\frac{(\pi-x)^3}{-3}\right)_0^{2\pi}}{2\pi}} = \sqrt{\frac{\frac{\pi^3}{3} + \frac{\pi^3}{3}}{2\pi}} \end{aligned}$$

$$= \sqrt{\frac{2\pi^2}{3 \cdot 2\pi}} = \sqrt{\frac{\pi^2}{3}}$$

$$\text{R.M.S} = \frac{\pi}{\sqrt{3}}$$

① Find the Fourier Series for $f(x) = x^2$ in $(-\pi, \pi)$
Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \dots = \frac{\pi^4}{90}$

Sol: W.K.T,
 $a_0 = \frac{2\pi^2}{3}$, $a_n = \frac{4}{n^2} (-1)^n$, $b_n = 0$.

Parseval's identity.

$$\frac{2}{b-a} \int_a^b [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{2}{2\pi} \int_{-\pi}^{\pi} x^4 dx = \frac{1}{\pi} \frac{\pi^5}{5} + \sum_{n=1}^{\infty} \frac{16}{n^4} (-1)^{2n}$$

$$\frac{1}{1^4} - \frac{2\pi^4}{5} + \frac{2\pi^4}{9} = \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{18\pi^4 - 10\pi^4}{45} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{8\pi^4}{45 \times 16} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

② Find the half range cosine series of the function $f(x) = x(\pi-x)$ in $0 < x < \pi$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$.

Sol:

W.K.T, $a_0 = \frac{\pi^2}{3}$, $a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{n^2} & \text{if } n \text{ is even} \end{cases}$

$b_n = 0$.

By parseval's identity,

$$\frac{2}{b-a} \int_a^b [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{2}{\pi} \int_0^{\pi} (x(\pi-x))^2 dx = \frac{\pi^4}{2 \cdot 9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{2}{\pi} \int_0^{\pi} (x^2(\pi^2 + x^2 - 2x\pi)) dx = \frac{\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$



$$\frac{1}{\sqrt{5}} \frac{2\pi^4}{5} - \frac{2}{9} \pi^4 = \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{18\pi^4 - 10\pi^4}{45} = 16 \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{8\pi^4}{45 \times 16} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}$$

② Find the half range cosine series of the function $f(x) = x(\pi-x)$ in $0 < x < \pi$. Hence deduce that $\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \dots = \frac{\pi^4}{90}$.

Sol:

$$\text{W.K.T, } a_0 = \frac{\pi^2}{3}, \quad a_n = \begin{cases} 0 & \text{if } n \text{ is odd} \\ -\frac{4}{n^2} & \text{if } n \text{ is even} \end{cases}$$

$$b_n = 0.$$

By Parseval's identity,

$$\frac{2}{b-a} \int_a^b [f(x)]^2 dx = \frac{a_0^2}{2} + \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

$$\frac{2}{\pi} \int_0^{\pi} (x(\pi-x))^2 dx = \frac{\pi^4}{2 \cdot 9} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$

$$\frac{2}{\pi} \int_0^{\pi} (x^2(\pi^2 + x^2 - 2x\pi)) dx = \frac{\pi^4}{18} + \sum_{n=1}^{\infty} \frac{16}{n^4}$$



$$\frac{2}{l} \left(\frac{x^3}{3} \right)_0^l = \frac{l^2}{2} + \sum_{n=\text{odd}} \frac{16l^2}{n^4 \pi^4} \left[\frac{\pi}{2} + \frac{2\pi}{3} \right] \frac{2}{\pi}$$
$$\frac{2}{l} \cdot \frac{l^3}{3} = \frac{l^2}{2} + \frac{16l^2}{\pi^4} \sum_{n=\text{odd}} \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right]$$
$$\frac{2l^2}{3} - \frac{l^2}{2} = \frac{16l^2}{\pi^4} \left[\frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots \right]$$
$$\frac{l^2}{6} = 16l^2$$
$$\frac{l^2}{6} \frac{\pi^4}{16l^2} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$
$$\frac{\pi^4}{96} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} + \dots$$