**TOPIC : 5 – ODD AND EVEN FUNCTIONS**

Even function:

If $f(x)$ is an even function, then

$f(x) = f(-x) \Rightarrow b_n = 0$.

Ex: $\cos x$, $|x|$, $x^2$.

$f(x) = x^2 \Rightarrow f(-x) = (-x)^2 = x^2 = f(x)$

$\therefore f(x) = f(-x)$.

$\therefore f(x)$ is an even function $\Rightarrow b_n = 0$.

Odd function:

If $f(x)$ is an odd function, then

$-f(x) = +f(-x)$.

Ex: $f(x) = x^3$, $\sin x$, $x^3$, $x\cos x$.

$\Rightarrow f(-x) = (-x)^3 = -x^3 = -f(x)$

$\therefore f(-x) = -f(x)$

$\therefore f(x)$ is an odd function $\Rightarrow a_0 = 0$ & $a_n = 0$

Problems based on $(-\pi, \pi)$ & $(-l, l)$

First check whether the function is odd or even.

If the function is even using the fourier formula in the interval $(-\pi, \pi)$.

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$ Here $b_n = 0$.

If the function is odd using the fourier formula in the interval $(-\pi, \pi)$.

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where $$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx \, dx$$

Here $a_0 = 0$ & $a_n = 0$.

If the function is even using the fourier formula in the interval $(-l, l)$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}$$

where $$a_0 = \frac{2}{b-a} \int_{-l}^{l} f(x) \, dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos \frac{n\pi x}{l} \, dx.$$

If the function is odd using the function fourier formula in the interval $(-l, l)$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{b-a} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} \, dx.$$

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{8}$$

<u>Sol:</u>

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$$f(-x) = f(x).$$

$\therefore f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2x^2}{2\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi}\right] = \cancel{\frac{2}{\pi}\cdot 2\pi}$$

$$a_0 = 0.$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos nx\,dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi}\left(1-\frac{2x}{\pi}\right)\cos nx\,dx$$

Here $u = 1 - \frac{2x}{\pi}$ $\qquad \int dv = \int \cos nx\,dx$

$u_1 = -\frac{2}{\pi}$ $\qquad v = \frac{\sin nx}{n}$

$u_2 = 0$ $\qquad v_1 = -\frac{\cos nx}{n^2}$

$$a_n = \frac{2}{\pi}\left[\left(1-\frac{2x}{\pi}\right)\frac{\sin nx}{n} - \frac{2\cos nx}{\pi n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left(1-\frac{2\pi}{\pi}\right)\frac{\sin n\pi}{n} - \frac{2\cos n\pi}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$= \frac{2}{\pi}\left[\frac{-2(-1)^n}{\pi n^2} + \frac{2}{\pi n^2}\right]$$

$$a_n = \frac{4}{\pi n^2}\left[1-(-1)^n\right]$$

$$a_n = \begin{cases} \frac{8}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \sum_{n=odd}^{\infty} \frac{8}{\pi n^2} \cos nx.$$

$$1 = \frac{8}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\cos nx.$$

$$\frac{\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$f(x) = |\cos x|.$

and deduce that

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi}{8}$$

Sol:

$$f(-x) = \begin{cases} 1 - \frac{2x}{\pi}, & -\pi \le -x \le 0 \\ 1 + \frac{2x}{\pi}, & 0 \le -x \le \pi \end{cases}$$

$$= \begin{cases} 1 - \frac{2x}{\pi}, & 0 \le x \le \pi \\ 1 + \frac{2x}{\pi}, & -\pi \le x \le 0 \end{cases}$$

$$f(-x) = f(x).$$

$\therefore f(x)$ is even $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a}\int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi}\left(1 - \frac{2x}{\pi}\right)dx$$

$$= \frac{2}{\pi}\left[x - \frac{2x^2}{2\pi}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\pi - \frac{\pi^2}{\pi}\right] = \cancel{\frac{2}{\pi}\cdot 2\pi}$$

$$a_0 = 0.$$

$$a_n = \frac{2}{\ } \int^{b} f(x)\cos nx\,dx$$

$f(x)$ is even. $\Rightarrow b_n = 0$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx.$$

$$a_0 = \frac{2}{b-a}\int_{-\pi}^{\pi} f(x)\,dx$$

$$= \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x|\,dx = \frac{2}{\pi}\int_0^{\pi} |\cos x|\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x\,dx + \int_{\frac{\pi}{2}}^{\pi} (-\cos x)\,dx\right\}$$

$$= \frac{2}{\pi}\left\{(\sin x)_0^{\frac{\pi}{2}} - (\sin x)_{\pi/2}^{\pi}\right\}$$

$$= \frac{2}{\pi}\left\{\sin\frac{\pi}{2} - \sin 0 - \sin\pi + \sin\frac{\pi}{2}\right\}$$

$$= \frac{2}{\pi}\{2\} = \frac{4}{\pi}$$

$$a_0 = \frac{4}{\pi}$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} |\cos x| \cos nx\,dx$$

$$= \frac{2}{\pi}\left\{\int_0^{\pi/2} \cos x \cos nx\,dx - \int_{\pi/2}^{\pi} \cos x \cos nx\,dx\right\}$$

$$a_n = \frac{1}{\pi}\left[\left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_0^{\pi/2} - \left(\frac{\sin(n+1)x}{n+1} + \frac{\sin(n-1)x}{n-1}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1} + \frac{\sin(n+1)\pi/2}{n+1} + \frac{\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{1}{\pi}\left[\frac{2\sin(n+1)\pi/2}{n+1} + \frac{2\sin(n-1)\pi/2}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\sin n\pi/2\cos\pi/2 + \cos n\pi/2\sin\pi/2}{n+1} + \frac{\sin\frac{n\pi}{2}\cos\frac{\pi}{2} - \cos\frac{n\pi}{2}\sin\frac{\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\left[\frac{\cos\frac{n\pi}{2}}{n+1} - \frac{\cos\frac{n\pi}{2}}{n-1}\right]$$

$$= \frac{2}{\pi}\frac{\cos n\pi/2}{n^2-1}\left[\cancel{n}-1-\cancel{n}-1\right]$$

$$a_n = \frac{-4}{\pi}\frac{\cos n\pi/2}{n^2-1}.$$

$$a_1 = \frac{2}{\pi}\left\{\int_0^{\pi/2}\cos x\cos x\,dx - \int_{\pi/2}^{\pi}\cos x\cos x\,dx\right\}$$

$$= \frac{2}{\pi}\int_0^{\pi/2}\left(\frac{1+\cos 2x}{2}\right)dx - \int_{\pi/2}^{\pi}\left(\frac{1+\cos 2x}{2}\right)dx$$

$$a_1 = \frac{2}{\pi}\left[\left(\frac{1}{2}x + \frac{\sin 2x}{4}\right)_0^{\pi/2} - \left(\frac{1}{2}x + \frac{\sin 2x}{4}\right)_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{1}{2}\cdot\frac{\pi}{2} + \frac{\sin 2\pi/2}{4} - \frac{1}{2}\pi - \frac{\sin 2\pi}{4} + \frac{1}{2}\cdot\frac{\pi}{2} + \frac{\sin 2\pi/2}{4}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{4} - \frac{\pi}{2} + \frac{\pi}{4}\right]$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{4} - \frac{\pi}{2}\right]$$

$$a_1 = 0.$$

$$f(x) = \frac{4}{\pi \cdot 2} + \sum_{n=2}^{\infty} \frac{-4}{\pi}\cdot\frac{\cos n\pi/2}{n^2-1}$$

$$f(x) = \frac{2}{\pi} - \frac{4}{\pi}\sum_{n=2}^{\infty}\frac{\cos n\pi/2}{n^2-1}.$$

Problems based on odd function:

1) Determine the fourier series for the functi

$$f(x) = \begin{cases} -1+x, & -\pi < x < 0 \\ 1+x, & 0 < x < \pi \end{cases}$$ & hence dedu

Given $f(x) = \begin{cases} x-1 & -\pi < x < 0 \\ 1+x & 0 < x < \pi \end{cases}$

$$f(-x) = \begin{cases} -x-1 & -\pi < -x < 0 \\ 1-x & 0 < -x < \pi \end{cases}$$

$$= -\begin{cases} x+1 & 0 < x < \pi \\ x-1 & -\pi < x < 0 \end{cases}$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function $\Rightarrow a_0 = 0, a_n = 0$

$$\therefore f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

where $b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$

$$= \frac{2}{\pi}\int_0^{\pi} x+1 \ \sin nx \, dx$$

$$= \frac{2}{\pi}\left[-(x+1)\frac{\cos nx}{n} + \frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-(\pi+1)\frac{\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{1}{n}\right]$$

$$= \frac{2}{\pi}\left[-(\pi+1)\frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$b_n = \frac{2}{n\pi}\left[1 - (1+\pi)(-1)^n\right]$$

2. Prove that $\frac{x(\pi^2-x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} - \frac{\sin 3x}{3^3} + \cdots$ in the interval $(-\pi, \pi)$.

Sol:

Let $f(x) = \frac{x(\pi^2-x^2)}{12}$

$$f(-x) = (-x)\left(\frac{\pi^2-(-x)^2}{12}\right)$$

$$= -x\left(\frac{\pi^2-x^2}{12}\right)$$

$$f(-x) = -f(x)$$

$f(x)$ is an odd function $\Rightarrow a_0 = 0$ & $a_n = 0$

Let the required fourier series be

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx.$$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} \frac{x(\pi^2-x^2)}{12}\sin nx\,dx.$$

$u = x\pi^2 - x^3$ $\int dv = \int \sin nx$

$u_1 = \pi^2 - 3x^2$ $v = \frac{-\cos nx}{n}$

$u_2 = -6x$ $v_1 = \frac{-\sin nx}{n^2}$

$u_3 = -6$ $v_2 = \frac{\cos nx}{n^3}$

$$= \frac{2}{12\pi}\int_0^{\pi}(\pi^2 x - x^3)\sin nx\,dx$$

$$= \frac{1}{6\pi}\left[-(\pi^2 x - x^3)\frac{\cos nx}{n} + (\pi^2-3x^2)\frac{\sin nx}{n^2} - 6x\frac{\cos nx}{n^3} + 6\frac{\sin nx}{n^4}\right]_0^{\pi}$$

$$b_n = -\frac{\cos n\pi}{n^3}$$

$$= -\frac{(-1)^n}{n^3}$$

$$\therefore f(x) = -\sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin nx.$$

$$\frac{x(\pi^2 - x^2)}{12} = \frac{\sin x}{1^3} - \frac{\sin 2x}{2^3} + \frac{\sin 3x}{3^3} + \cdots$$

(3) $f(x) = x^2 + x$ in $(-\pi, \pi)$ & $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = x + x^2$$

$$f(-x) = -x + (-x)^2$$

$$= -x + x^2$$

$$f(x) \neq f(-x)$$

$$f(x) = -f(x)$$

$\therefore f(x)$ is neither even nor odd.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx$$

$$= \frac{1}{\pi} \int_{-\pi}^{\pi} x + x^2\, dx$$

$$= \frac{1}{\pi} \left( \frac{x^2}{2} + \frac{x^3}{3} \right)_{-\pi}^{\pi}$$

$$= \frac{1}{\pi} \left[ \frac{\pi^2}{2} + \frac{\pi^3}{3} - \frac{\pi^2}{2} + \frac{\pi^3}{3} \right]$$

$$a_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{\pi} x+x^2 \cos nx\,dx$$

$$= \frac{1}{\pi}\left[(x+x^2)\frac{\sin nx}{n} + (1+2x)\frac{\cos nx}{n^2} - 2\frac{\sin nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{\cos n\pi}{n^2} - (1-2\pi)\frac{\cos n(-\pi)}{n^2}\right]$$

$$= \frac{1}{\pi}\left[(1+2\pi)\frac{(-1)^n}{n^2} - (1-2\pi)\frac{(-1)^n}{n^2}\right]$$

$$= \frac{(-1)^n}{n^2\pi}\left[\cancel{1}+2\pi-\cancel{1}+2\pi\right]$$

$$= \frac{(-1)^n 4\pi}{n^2\pi}$$

$$a_n = \frac{4(-1)^n}{n^2}.$$

$$b_n = \frac{2}{2\pi}\int_{-\pi}^{\pi} x+x^2 \sin nx\,dx$$

$$= \frac{1}{\pi}\left[-(x+x^2)\frac{\cos nx}{n} + (1+2x)\frac{\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{\pi}\left[-(\pi+\pi^2)\frac{(-1)^n}{n} + \frac{2(-1)^n}{n^3} + (-\pi+\pi^2)\frac{\cos n(-\pi)}{n} - \frac{2\cos n(-\pi)}{n^3}\right]$$

$$b_n = \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} - \frac{\pi^2 \cancel{(-1)^n}}{\cancel{n}} - \frac{\pi(-1)^n}{n} + \frac{\pi^2 \cancel{(-1)^n}}{\cancel{n}}\right]$$

$$= \frac{1}{\cancel{\pi}}\left[\frac{-2\cancel{\pi}(-1)^n}{n}\right]$$

$$b_n = \frac{2(-1)^{n+1}}{n}$$

$$b_n = \begin{cases} \frac{2}{n} & \text{if } n \text{ is odd} \\ -\frac{2}{n} & \text{if } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\cancel{2}\pi^2}{3\cdot\cancel{2}} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin nx$$

If $\cancel{x=0}$, $x=\pi$

$$\pi + \pi^2 = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^n}{n^2}\cos n\pi$$

$$\pi + \pi^2 - \frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

If $x=\pi$ (discontinuous & end point)

$$\frac{f(-\pi)+f(\pi)}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}(-1)^n$$

$$\frac{\cancel{-\pi}+\pi^2+\cancel{\pi}+\pi^2}{2} = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\pi^2 - \frac{\pi^2}{3} = 4\sum_{n=1}^{\infty}\frac{(-1)^{2n}}{n^2}$$

$$\frac{\cancel{2}\pi^2}{3\cdot\cancel{4}2} = \cancel{4}\cdot\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$