TOPIC : 2 – PROBLEMS BASED ON FULL RANGE SERIES

Dirichlet condition:

i) $f(x)$ is periodic, single valued and finite

ii) $f(x)$ has a finite no. of finite discontinuous.

iii) $f(x)$ has no infinite discontinuous

iv) $f(x)$ has a finite no. of maxima and minima

Formula for fourier series in $(0, 2\pi)$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

where
$$a_0 = \frac{2}{b-a} \int_a^b f(x)\, dx$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx\, dx$$

$$b_n = \frac{2}{b-a} \int_a^b f(x) \sin nx\, dx.$$

Problems:

① Expand $f(x) = x^2$ in $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2\,dx$$

$$= \frac{1}{\pi} \left(\frac{x^3}{3}\right)_0^{2\pi}$$

$$= \frac{8\pi^{\cancel{3}2}}{3\cancel{\pi}}$$

$$= \frac{8\pi^2}{3}.$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx\,dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} x^2 \cos nx\,dx.$$

$$= \frac{1}{\pi} \int_0^{2\pi} x^2 \cos nx\,dx.$$

Here $u = x^2$ $\quad \int dv = \int \cos nx$

$u_1 = 2x \quad V = \frac{\sin nx}{n}$

$u_2 = 2 \quad V_1 = -\frac{\cos nx}{n^2}$

$V_2 = -\frac{\sin nx}{n^3}$

$$a_n = \frac{1}{\pi}\left[x^2\frac{\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\frac{4\pi^2 \sin 2n\pi}{n} + \frac{4\pi\cos 2n\pi}{n^2} - \frac{2\sin 2n\pi}{n^3} - \frac{2\cdot 0\cos 2n\pi}{n^2}\right] \quad \left(\because \begin{array}{l}\sin 2n\pi = 0\\ \cos 2n\pi = 1.\end{array}\right)$$

$$= \frac{1}{\pi}\left[\frac{4\pi}{n^2}\right]$$

$$= \frac{4}{n^2}.$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x^2 \sin nx\,dx = \frac{1}{\pi}\int_0^{2\pi} x^2\sin nx\,dx$$

$u = x^2 \qquad \int dv = \int \sin nx$

$u_1 = 2x \qquad v = -\frac{\cos nx}{n}$

$u_2 = 2 \qquad v_1 = -\frac{\sin nx}{n^2} \qquad v_2 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi}\left[\frac{-x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + 2\frac{\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{4\pi^2\cos 2n\pi}{n} + \frac{4\pi\sin 2n\pi}{n^2} + \frac{2\cos 2n\pi}{n^3}\right.$$

$$b_n = \frac{1}{\pi}\left[\frac{-4\pi^2}{n} + \frac{2}{n^2} - \frac{2}{n^3}\right]$$

$$b_n = -\frac{4\pi}{n}.$$

$$\therefore f(x) = \frac{8\pi^2}{3\cdot 2} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin nx.$$

$$= \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}\cos nx + \sum_{n=1}^{\infty} \frac{-4\pi}{n}\sin nx.$$

Deduction:

Put $x=0$ [end points discontinuous]

$$\frac{f(0)+f(2\pi)}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{0+4\pi^2}{2} = \frac{4\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$2\pi^2 - \frac{4\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{6\pi^2 - 4\pi^2}{3} = 4\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2\pi^2}{3\cdot 4_2} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

(2) Expand $f(x) = (\pi - x)^2$ in $(0, 2\pi)$ and hence deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx.$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} (\pi - x)^2 dx$$

$$= \frac{1}{\pi} \left[-\frac{(\pi - x)^3}{3} \right]_0^{2\pi}$$

$$= \frac{1}{\pi} \left[+\frac{\pi^3}{3} + \frac{\pi^3}{3} \right] = \frac{1}{\pi} \frac{2\pi^3}{3}$$

$$= \frac{2\pi^2}{3}.$$

$$a_n = \frac{2}{b-a} \int_a^b f(x) \cos nx\,dx$$

$$= \frac{2}{2\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} (\pi - x)^2 \cos nx\,dx$$

$u = (\pi - x)^2$ $\qquad \int dv = \int \cos nx$

$u_1 = -2(\pi - x)$ $\qquad v = \frac{\sin nx}{n}$

$$a_n = \frac{1}{\pi}\left[(\pi-x)^2\frac{\sin nx}{n} - \frac{2(\pi-x)\cos nx}{n^2} - \frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{2(\pi-2\pi)\cos 2n\pi}{n^2} + \frac{2\pi\cos 2n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left(\frac{4\pi\cos 2n\pi}{n^2}\right) = \frac{1}{\pi}\left(\frac{4\pi}{n^2}\right)$$

$$= \frac{4}{n^2}.$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2\sin nx\,dx.$$

$u = (\pi-x)^2$ $\quad \int dv = \int \sin nx$

$u_1 = -2(\pi-x)$ $\quad v = -\frac{\cos nx}{n}$

$u_2 = 2.$ $\quad v_1 = -\frac{\sin nx}{n^2}$; $v_2 = \frac{\cos nx}{n^3}$

$$b_n = \frac{1}{\pi}\left[-\frac{(\pi-x)^2\cos nx}{n} - \frac{2(\pi-x)\sin nx}{n^2} + \frac{2\cos nx}{n^3}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-\frac{(\pi-2\pi)^2\cos 2n\pi}{n} + \frac{2\cos 2n\pi}{n^3} + \frac{\pi^2\cos 0}{n} - \frac{2\cos 0}{n^3}\right]$$

$$= \frac{1}{\pi}\left[-\cancel{\frac{\pi^2}{n}} + \cancel{\frac{2}{n^3}} + \cancel{\frac{\pi^2}{n}} - \cancel{\frac{2}{n^3}}\right]$$

$b_n = 0$.

$\therefore f(x) = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$

Deduction:

Put $x=0$ (end point & discontinuous)

$$\frac{f(0)+f(2\pi)}{2} = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{\pi^2+\pi^2}{2} = \frac{2\pi^2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\pi^2 - \frac{2\pi^2}{3} = \sum_{n=1}^{\infty} \frac{4}{n^2}$$

$$\frac{4\pi^2}{3\cdot 4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Problems on $(0, 2\pi)$:

1. Expand $f(x) = \begin{cases} x & 0 < x < \pi \\ 2\pi - x, & \pi < x < 2\pi \end{cases}$.

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}$$

Sol: $a_0 = \frac{1}{\pi}\left[\int_0^{\pi} x\,dx + \int_{\pi}^{2\pi} (2\pi - x)\,dx\right]$.

$$= \frac{1}{\pi}\left[\left(\frac{x^2}{2}\right)_0^{\pi} + \left(2\pi x - \frac{x^2}{2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi^2}{2} + 4\pi^2 - \frac{4\pi^2}{2} - 2\pi^2 + \frac{\pi^2}{2}\right]$$

$$= \frac{1}{\pi}\left(\pi^2\right)$$

$$a_0 = \pi.$$

$$a_n = \frac{2}{b-a}\int_a^b f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_0^{\pi} x\cos nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\cos nx\,dx\right]$$

$u = x \qquad \int dv = \int \cos nx\,dx \qquad u = (2\pi - x)$

$u_1 = 1 \qquad V = \frac{\sin nx}{n} \qquad u_1 = -1.$

$u_2 = 0 \qquad V_1 = -\frac{\cos nx}{n^2} \qquad u_2 = 0$

$$a_n = \frac{1}{\pi}\left[\left(\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{\sin nx}{n} - \frac{\cos nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\pi\sin n\pi}{n} + \frac{\cos n\pi}{n^2} - \frac{1}{n^2} - \frac{\cos 2n\pi}{n^2} + \frac{\cos n\pi}{n^2}\right]$$

$$= \frac{1}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2} - \frac{1}{n^2} + \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{1}{\pi}\left[\frac{2(-1)^n - 2}{n^2}\right]$$

$$= \frac{2}{n^2\pi}\left[(-1)^n - 1\right]$$

$$a_n = \begin{cases} \frac{-4}{\pi n^2} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even.} \end{cases}$$

$$b_n = \frac{2}{b-a}\int_a^b f(x)\sin nx\,dx.$$

$$= \frac{2}{2\pi}\left[\int_0^{\pi} x\sin nx\,dx + \int_{\pi}^{2\pi}(2\pi - x)\sin nx\,dx\right]$$

$$\int dv = \int \sin nx$$

$u = x$ $\qquad v = -\frac{\cos nx}{n}$ $\qquad u = (2\pi - x)$

$u_1 = 1$ $\qquad v_1 = -\frac{\sin nx}{n^2}$ $\qquad u_1 = -1.$

$u_2 = 0$ $\qquad u_2 = 0$

$$b_n = \frac{1}{\pi}\left[\left(\frac{-x\cos nx}{n} + \frac{\sin nx}{n^2}\right)_0^{\pi} + \left((2\pi - x)\frac{\cos nx}{n} - \frac{\sin nx}{n^2}\right)_{\pi}^{2\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi\cos n\pi}{n} + \frac{\sin n\pi}{n^2} + \frac{\pi\cos n\pi}{n}\right]$$

$$= \frac{1}{\pi}\left[\frac{-\pi(-1)^n}{n} + \frac{\pi(-1)^n}{n}\right]$$

$$= 0.$$

$$\therefore f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \cos nx.$$

Deduction:

Put $x=0$.

$$\pi = \frac{\pi}{2} + 4\sum_{n=odd}^{\infty} \frac{1}{\pi n^2}$$

$$\pi - \frac{\pi}{2} = \frac{4}{\pi} \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{2} \times \frac{\pi}{4} = \sum_{n=odd}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots = \frac{\pi^2}{8}.$$

② F.S $f(x) = x \sin x$ & hence deduce that

$$\frac{\pi-2}{4} = \frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots$$

Sol:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{2}{b-a} \int_a^b f(x)\,dx$$

$$= \frac{1}{\pi} \int_0^{2\pi} x \sin x \, dx$$

$u = x \qquad \int dv = \int \sin x$

$u_1 = 1 \qquad v = -\cos x$

$u_2 = 0 \qquad v_1 = -\sin x$

$$a_0 = \frac{1}{\pi}\left[-x\cos x + \sin x\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[-2\pi\cos 2\pi\right]$$

$$= \frac{1}{\pi}\left[-2\pi\right]$$

$$a_0 = -2.$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x\cos nx \sin x\,dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left[\sin(n+1)x - \sin(n-1)x\right]dx$$

$u = x \quad dv = \int \sin(n+1)x\,dx \quad \int dv = \int \sin(n-1)x\,dx$

$u_1 = 1 \quad v = -\frac{\cos(n+1)x}{n+1} \quad v = -\frac{\cos(n-1)x}{n-1}$

$V_1 = -\frac{\sin(n+1)x}{(n+1)^2} \quad V_1 = -\frac{\sin(n-1)x}{(n-1)^2}$

$$a_n = \frac{1}{2\pi}\left[-\frac{x\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2}\right]_0^{2\pi} - \left[\frac{-x\cos(n-1)x}{n-1} + \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[-\frac{2\pi\cos 2(n+1)\pi}{n+1} + \frac{2\pi}{n-1}\right]$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right]$$

$$= \frac{2\pi}{2\pi}\left[\frac{-n+1+n+1}{n^2-1}\right]$$

$$= \frac{2}{n^2-1} \quad (n \neq 1)$$

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx$$

$$= \frac{1}{\pi}\int_0^{2\pi} x \frac{\sin 2x}{2} \, dx$$

$u = x \qquad dv = \int \sin 2x \, dx$

$u_1 = 1 \qquad v = -\frac{\cos 2x}{2}$

$u_2 = 0 \qquad v_1 = -\frac{\sin 2x}{4}$

$$a_1 = \frac{1}{2\pi}\left[-\frac{x\cos 2x}{2} + \frac{\sin 2x}{4}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi \cos 2\pi}{2}\right]$$

$$= \cancel{\frac{1}{2}}$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x\left(\cos(n-1)x - \cos(n+1)x\right)dx$$

$$= \frac{1}{2\pi}\left[\int_0^{2\pi} x\cos(n-1)x \, dx - \int_0^{2\pi} x \cos(n+1)x \, dx\right]$$

$$= \frac{1}{2\pi}\left[\left(x\frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2}\right)_0^{2\pi} - \left(\frac{x \sin(n+1)x}{\ldots} + \frac{\cos(n+1)x}{(n+1)^2}\right)_0^{2\pi}\right\}$$

$$b_n = \frac{1}{2\pi}\left[\cancel{\frac{1}{(n-1)^2}} \mp \cancel{\frac{1}{(n+1)^2}} - \cancel{\frac{1}{(n-1)^2}} + \cancel{\frac{1}{(n+1)^2}}\right]$$

$$b_n = 0.$$

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x\sin x \, \sin x \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\, dx$$

$$= \frac{1}{2\pi}\left[x\left(x-\frac{\sin 2x}{2}\right) - \left(\frac{x^2}{2}+\frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \cancel{\frac{1}{4}} + \cancel{\frac{1}{4}}\right]$$

$$= \frac{1}{2\pi}\left[4\pi^2 - \frac{4\pi^2}{2}\right] = \frac{1}{2\pi}\cdot\frac{4\pi^2}{2}$$

$$b_1 = \pi.$$

$$\therefore \quad f(x) = -\frac{2}{2} + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx - \frac{1}{2}\cos x + \pi\sin x$$

$$= -1 - \frac{1}{2}\cos x + \pi\sin x + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos nx.$$

Deduction:

put $x = \frac{\pi}{2}$

$$\frac{\pi}{2}\sin\frac{\pi}{2} = -1 + \pi + 2\sum_{n=2}^{\infty}\frac{1}{n^2-1}\cos\frac{n\pi}{2}$$

$$\frac{\pi}{2} + 1 - \pi = 2\sum_{n=2}^{\infty}\frac{1}{(n+1)(n-1)}\cos\frac{n\pi}{2}$$

$$\left(1-\frac{\pi}{2}\right)\cdot\frac{1}{2} = \sum_{n=2}^{\infty} \frac{1}{(n+1)(n-1)} \cos\frac{n\pi}{2}$$

$$\frac{1}{2}-\frac{\pi}{4} = \frac{1}{1\cdot 3}\cos\pi + \frac{1}{4\cdot 2}\cos\frac{3\pi}{2} + \frac{1}{5\cdot 3}\cos 2\pi + \cdots$$

$$\frac{1}{2}-\frac{\pi}{4} = \frac{-1}{1\cdot 3} + \frac{1}{3\cdot 5} - \frac{1}{5\cdot 7} + \cdots$$

$$\frac{1}{1\cdot 3} - \frac{1}{3\cdot 5} + \frac{1}{5\cdot 7} - \cdots = \frac{\pi-2}{4}.$$