Method of Multipliers:

choose any three multipliers l, m, n which may be constants or functions of x, y, z, we have

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l\,dx + m\,dy + n\,dz}{lP + mQ + nR}$$

It is possible to choose l, m, n such that $lP + mQ + nR = 0$ then $l\,dx + m\,dy + n\,dz = 0$

If $l\,dx + m\,dy + n\,dz$ is an exact differential then on integration we get a solution $u = a$.

The multipliers l, m, n are called Lagrangian multipliers.

(1) Solve: $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

Sol:

Given $x(y^2-z^2)p+y(z^2-x^2)q=z(x^2-y^2)$

This is of the form

$$Pp+Qq=R.$$

where $P=x(y^2-z^2)$ $Q=y(z^2-x^2)$ $R=z(x^2-y^2)$

The subsidiary eqns are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)}=\frac{dy}{y(z^2-x^2)}=\frac{dz}{z(x^2-y^2)}$$

$$\text{Each ratio}=\frac{x\,dx+y\,dy+z\,dz}{x^2(y^2-z^2)+y^2(z^2-x^2)+z^2(x^2-y^2)}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{x^2y^2-x^2z^2+y^2z^2-y^2x^2+z^2x^2-z^2y^2}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0$$

$$\Rightarrow \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c.$$

$$\text{Each ratio}=\frac{\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$\frac{dx}{x}+\frac{dy}{y}+\frac{dz}{z}=0$$

$$\log x+\log y+\log z=\log c_1$$

2. Solve: $(mz-ny)p+(nx-lz)q=ly-mx$

Sol:
Given $(mz-ny)p+(nx-lz)q=ly-mx$

This eqn is of the form $Pp+Qq=R$

where $P=mz-ny$, $Q=nx-lz$, $R=ly-mx$

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$

$$\frac{dx}{mz-ny}=\frac{dy}{nx-lz}=\frac{dz}{ly-mx}$$

$$\text{Each ratio}=\frac{dx+dy+dz}{mz-ny+nx-lz+ly-mx}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{xmz-xny+nyx-ylz+lyz-mxz}$$

$$\Rightarrow x\,dx+y\,dy+z\,dz=0.$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c.$$

$$\text{Each ratio}=\frac{l\,dx+m\,dy+n\,dz}{lmz-lny+mnx-mlz+lny-mnx}$$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

Integrating

$$lx + my + nz = c_1$$

The general solution is

$$\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ lx+my+nz\right)=0.$$

(8) Solve: $(3z-4y)p+(4x-2z)q = 2y-3x$

Sol:
This eqn is of the form $Pp+Qq=R$

$P = 3z-4y$, $Q = 4x-2z$, $R = 2y-3x$

$$\frac{dx}{3z-4y} = \frac{dy}{4x-2z} = \frac{dz}{2y-3x}$$

$$\text{Each ratio} = \frac{x\,dx+y\,dy+z\,dz}{3xz-4yx+4xy-2yz+2yz-3xz}$$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2} = c$$

$$\text{Each ratio} = \frac{2\,dx+3\,dy+4\,dz}{6z-8y+12x-6z+8y-12x}$$

$\Rightarrow 2\,dx + 3\,dy + 4\,dz = 0.$

$2x+3y+4z=c_2$.

The general solution is

$$\phi\left(\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2},\ 2x+3y+4z\right)=0.$$

(4) Solve: $(y-xz)p+(yz-x)q=(x+y)(x-y)$

Sol:

This eqn is of the form $Pp+Qq=R$

$P=y-xz,\quad Q=yz-x,\quad R=(x+y)(x-y)$

$$\frac{dx}{y-xz}=\frac{dy}{yz-x}=\frac{dz}{(x+y)(x-y)}=\frac{dz}{x^2-y^2}$$

$$\text{Each ratio}=\frac{xdx+ydy+zdz}{xy-x^2z+y^2z-xy+x^2z-zy^2}$$

$$\Rightarrow xdx+ydy+zdz=0$$

Integrating,

$$\frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}=c$$

$$\Rightarrow u=\frac{x^2}{2}+\frac{y^4}{2}+\frac{z^2}{2}.$$

$$\text{Each ratio}=\frac{ydx+xdy+dz}{y^2-xyz+xyz-x^2+x^2-y^2}$$

$$ydx+xdy+dz=0.$$

Integrating, $\cancel{y}x+\cancel{x}y+\cancel{z}=$

$$d(xy) + dz = 0$$

on integration, we get

$$xy + z = c_1$$

$$v = xy + z.$$

The general solution is

$$\phi\left(\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2},\ xy + z\right) = 0$$