



① Lagrange's Linear Equation:
Method of Grouping:
In the auxiliary equation
$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$
 if the variables can be separated in any pair of equations, then we get a solution of the form
 $u(x, y) = a \quad \& \quad v(x, y) = b.$

① solve: $px + qy = z$

sol: Given $px + qy = z$
 $p = x, \quad q = y, \quad r = z$

The subsidiary equations are

$$\frac{dx}{p} = \frac{dy}{q} = \frac{dz}{r}$$

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c_1$$

$$x = (y \cdot c_1)$$

$$\Rightarrow \frac{x}{y} = c_1 \quad (i) \quad u = \frac{x}{y}$$

$$\frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c_2$$

$$y = z c_2$$

$$\Rightarrow \frac{y}{z} = c_2 \quad (ii) \quad v = \frac{y}{z}$$

The solution of the given p.d.e is

$$\phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0.$$



② Write the general integral of

$$pyz + qzx = xy.$$

Sol:

The given eqn is of the form

$$Pp + Qq = R.$$

$$P = yz, \quad Q = zx, \quad R = xy$$

$$\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}$$

$$\frac{dx}{yz} = \frac{dy}{zx} \Rightarrow x dx = y dy$$

$$\Rightarrow \int x dx = \int y dy$$

$$\Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + c_1$$

$$\Rightarrow c_1 = x^2 - y^2$$

$$\frac{dy}{zx} = \frac{dz}{xy} \Rightarrow y dy = z dz$$

$$\Rightarrow \frac{y^2}{2} = \frac{z^2}{2} + c_2$$

$$\Rightarrow c_2 = y^2 - z^2$$

$$\therefore \phi(x^2 - y^2, y^2 - z^2) = 0.$$

③ Solve: $x(y-z)p + y(z-x)q = z(x-y)$

Sol:

The given eqn is of the form

$$Pp + Qq = R$$

$$P = x(y-z), \quad Q = y(z-x), \quad R = z(x-y)$$



$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)}$$

$$\frac{dx+dy+dz}{xy-xz+yz-xy+zx-zy} = \text{Each ratio}$$

$$dx+dy+dz = 0$$

Integrating, $x+y+z = C_1$

$$\text{Each ratio} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{y-z + z-x + x-y}$$

$$\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0$$

Integrating $\log x + \log y + \log z = \log C_2$

$$\log (xyz) = \log C_2$$

$$xyz = C_2$$

$\therefore \phi(x+y+z, xyz) = 0$

(A) $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Sol: This eqn is of the form $Pp + Qq = R$

$$P = x^2-yz \quad Q = y^2-zx \quad R = z^2-xy$$

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy}$$

$$\frac{dx+dy+dz}{x^2-yz + y^2-zx + z^2-xy} = \frac{x dx + y dy + z dz}{x^3 - xyz + y^3 - yxz + z^3 - xyz}$$



$$\frac{x dx + y dy + z dz}{x^3 + y^3 + z^3 - 3xyz} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx}$$

$$\frac{x dx + y dy + z dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} = \frac{dx + dy + dz}{x^2+y^2+z^2-xy-yz-zx}$$

$$x dx + y dy + z dz = (x+y+z)(dx+dy+dz)$$

Integrating,

$$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + C$$

$$x^2 + y^2 + z^2 = (x+y+z)^2 + C$$

$$x^2 + y^2 + z^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx + C$$

$$-2(xy + yz + zx) = C$$

$$xy + yz + zx = u \text{ (constant)}$$

$$\frac{dx - dy}{x^2 - yz - y^2 + zx} = \frac{dy - dz}{y^2 - zx - z^2 + xy}$$

$$\frac{d(x-y)}{(x^2 - y^2) + z(x-y)} = \frac{d(y-z)}{y^2 - z^2 + x(y-z)}$$

$$\frac{d(x-y)}{(x-y)(x+y+z)} = \frac{d(y-z)}{(y-z)(y+z+x)}$$



Integrating on both sides, we get

$$\log(x-y) = \log(y-z) + \log c$$
$$\log(x-y) - \log(y-z) = \log c$$
$$\log \left(\frac{x-y}{y-z} \right) = \log c$$
$$\frac{x-y}{y-z} = c$$

The general solution is

$$\phi \left(xy + yz + zx, \frac{x-y}{y-z} \right) = 0.$$