Unit - I ①

Partial Differential Equations

Formation of Partial Differential equations by Elimination of arbitrary constants

Consider an equation $f(x, y, z, a, b) = 0$ — ①

where a & b denote arbitrary constants.

A p.d.e is formed by eliminating the arbitrary constants that occur in the functional relation between the variables.

Using $\frac{\partial z}{\partial x} = p$. $\frac{\partial z}{\partial y} = q$.

1. Form the p.d.e by eliminating the arbitrary constants a & b form $z = ax + by$ → ①

Diff p. w.r.to x we get

$$\frac{\partial z}{\partial x} = a \Rightarrow p = a.$$

Diff p. w.r. to y we get

$$\frac{\partial z}{\partial y} = b \Rightarrow q = b$$

∴ Eqn ① becomes, $z = px + qy$.

2. Eliminate the arbitrary constants a & b form $z = (x^2+a)(y^2+b)$.

Sol:

$$z = (x^2+a)(y^2+b)$$

Diff p.w.r.to x.

$$p = \frac{\partial z}{\partial x} = 2x(y^2+b) \Rightarrow \frac{p}{2x} = y^2+b$$

Diff p.w.r.to y.

$$q = 2y(x^2+a) \Rightarrow \frac{q}{2y} = x^2+a$$

$\therefore$ Eqn ① becomes,

$$z = \frac{p}{2x} \cdot \frac{q}{2y}$$

$$4xyz = pq$$

3. $z = a(x+y)+b$.

Sol:

$$z = a(x+y)+b$$

Diff w.r.to x.

$$p = a \rightarrow ①$$

Diff w.r.to y.

$$q = a \rightarrow ②$$

From ① & ② we get $pq = a$. $p = q$.

4. Find the partial differential equation of all planes having equal intercepts on the x & y axis

Sol: Intercept form of the plane equation is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$$

Given $a=b$ (Equal intercepts on the x & y axis)

$$\frac{x}{a}+\frac{y}{a}+\frac{z}{c}=1.$$

Diff p.w.r.to x, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \frac{1}{a}=\frac{-1}{c}\frac{\partial z}{\partial x}=\frac{-1}{c}p \rightarrow (1)$$

Diff p.w.r. to y, we get

$$\frac{1}{a}+\frac{1}{c}\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \frac{1}{a}=-\frac{1}{c}\frac{\partial z}{\partial y}=-\frac{1}{c}q \rightarrow (2)$$

From (1) & (2),

$$-\frac{1}{c}p=-\frac{1}{c}q$$

$$p=q.$$

6. $(x-a)^2+(y-b)^2+z^2=1.$

Sol:

Diff w.r.to x, we get

$$2(x-a)+2z\frac{\partial z}{\partial x}=0$$

$$2(x-a)=-2zp$$

$$x-a=-zp \rightarrow (1)$$

Diff p.w.r.to y, we get

$$2(y-b)+2z\frac{\partial z}{\partial y}=0$$

$$2(y-b)=-2zq$$

$$y-b=-zq \rightarrow (2)$$

Using (1) & (2), we get

$$(-zp)^2+(-zq)^2+z^2=1$$

$$z^2p^2+z^2q^2+z^2=1$$

$$z^2(p^2+q^2+1)=1.$$

(6) $z = (x+a)^3 + (y-b)^2$

Sol:

Diff p.w.r.to x, we get

$$\frac{\partial z}{\partial x} = 3(x+a)^2 \cdot 1$$

$$\Rightarrow \frac{p}{3} = (x+a)^2$$

Diff p.w.r.to y, we get

$$\frac{\partial z}{\partial y} = 2(y-b)$$

$$\frac{q}{2} = y-b.$$

$$z = \left(\frac{p}{3}\right)^{3/2} \left(\frac{q}{2}\right)^2 \quad z = \left(\frac{p}{3}\sqrt{\frac{p}{3}}\right)^3 + \left(\frac{q}{2}\right)^2$$

$$z = \left(\frac{p^{3/2}}{3^{3/2}}\right)^3 + \left(\frac{q}{2}\right)^2$$

(7) $(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha.$

Sol:

$$(x-a)^2 + (y-b)^2 = z^2 \cot^2 \alpha$$

Diff w.r. to x we get

$$\cancel{2}(x-a) = \cancel{2} z p \cot^2 \alpha$$

$$x - a = z p \cot^2 \alpha$$

Diff w.r. to y, we get

$$\cancel{2}(y-b) = \cancel{2} z q \cot^2 \alpha$$

$$y - b = z q \cot^2 \alpha$$

$$z^2 p^2 \cot^4 \alpha + z^2 q^2 \cot^4 \alpha = z^2 \cot^2 \alpha$$

$$\div z^2 \cot^2 \alpha, \quad p^2 \cot^2 \alpha + q^2 \cot^2 \alpha = 1.$$