Rules in Quantifiers

Rule US : (Universal Specification) :

From $(x) A(x)$ one can conclude $A(y)$

If a statement of the form $(\forall x) A(x)$ is assumed to be true, then the universal quantifier can be dropped to obtain $A(y)$ is true for any arbitrary object 'y' in the universe.

Rule ES : [Existential Specification]

From $(\exists x) A(x)$ one can conclude $A(y)$

Provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time ES is used

Note

[The conditions of ES are more restrictive than ordinarily required, but they ~~don~~ ~~not~~ do not affect the possibility of deriving any conclusion]

Rule EG : [Existential Generation]

From $A(x)$ one can conclude $(\exists y) A(y)$.

If $A(x)$ is true for some element x in the universe, then $(\exists y) A(y)$ is true

Rule UG: [universal generalization]

From $A(x)$ one can conclude $(y)A(y)$ provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from the use of ES, then no variables introduced by that use of ES appear free in $A(x)$

1) Show that $(x)(H(x) \to M(x)) \wedge H(s) \Rightarrow M(s)$.

This problem is a symbolic translation of a well-known argument known as the "Socrates argument" which is given by

All men are mortal.
Socrates is a man
Therefore Socrates is a mortal.

If we denote $H(x)$: x is a man, $M(x)$: x is a mortal and s: Socrates, we can put the argument in the above form.

Soln:

Steps	derivation	rule	reason
1.	$(x)(H(x) \to M(x))$	P	Given premise.
2.	$H(s) \to M(s)$	US, (1)	
3.	$H(s)$	P	Given Premise.
4.	$M(s)$	T	(2),(3) $P, P \to Q \Rightarrow Q$

2) Show that $(x)(P(x) \rightarrow Q(x)) \wedge (x)(Q(x) \rightarrow R(x)) \Rightarrow (x)(P(x) \rightarrow R(x))$

Soln:

Steps	derivation	rule	Reason
1	$(x)(P(x) \rightarrow Q(x))$	P	Given
2.	$P(y) \rightarrow Q(y)$	US, (1)	
3.	$(x)(Q(x) \rightarrow R(x))$	P	Given
4.	$Q(y) \rightarrow R(y)$	US, (3)	
5.	$P(y) \rightarrow R(y)$	T	(2),(4) Hypothetical syllogism. $(P \rightarrow Q) \wedge (Q \rightarrow R) \Rightarrow (P \rightarrow R)$
6.	$(x)(P(x) \rightarrow R(x))$	UG, (5)	

3) Show that $(\exists x) M(x)$ follows logically from the premises $(x)(H(x) \rightarrow M(x))$ & $(\exists x) H(x)$

Soln:

Steps	derivation	rule	reason
1.	$(\exists x) H(x)$	P	Given
2.	$H(y)$	ES, (1)	
3.	$(x)(H(x) \rightarrow M(x))$	P	Given
4.	$H(y) \rightarrow M(y)$	US (1)	
5.	$M(y)$	T	(2),(4) $P, P \rightarrow Q \Rightarrow Q$
6.	$(\exists x) M(x)$	EG (5)	

A) Prove that

$(\exists x)(P(x) \wedge Q(x)) \Rightarrow (\exists x) P(x) \wedge (\exists x) Q(x)$ but the converse not true.

Solu:

Steps	derivation	rule	reason
(1)	$(\exists x)(P(x) \wedge Q(x))$	P	Given
(2)	$P(y) \wedge Q(y)$	ES, (1)	y fixed.
(3)	$P(y)$	T	(2) $P \wedge Q \Rightarrow P$
(4)	$Q(y)$	T	(2) $P \wedge Q \Rightarrow Q$
(5)	$(\exists x) P(x)$	EG, (3)	
(6)	$(\exists x) Q(x)$	EG, (4)	
(7)	$(\exists x) P(x) \wedge (\exists x) Q(x)$	T	(5),(6) $P, Q \Rightarrow P \wedge Q$.

It is instructive to try to prove the converse which does not hold. The derivation is

(1)	$(\exists x) P(x) \wedge (\exists x) Q(x)$	P	Given
(2)	$(\exists x) P(x)$	T	(1), $P \wedge Q \Rightarrow P$
(3)	$(\exists x) Q(x)$	T	(1), $P \wedge Q \Rightarrow Q$
(4)	$P(y)$	ES	(2)
(5)	$Q(z)$	ES	(3)

Note that in Step 4, y is fixed and it is no longer possible to use that variable again in Step 5.

(5) Show that from

(a) $(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$

(b) $(\exists y)(M(y) \wedge \neg W(y))$

the conclusion $(x)(F(x) \rightarrow \neg S(x))$ follows.

Soln:

Steps	derivation	rule	reason
1	$(\exists y)(M(y) \wedge \neg W(y))$	P	Given
2.	$M(z) \wedge \neg W(z)$	ES,(1)	z fixed
3.	$\neg(M(z) \rightarrow W(z))$	T(2),	$\neg(P \rightarrow Q) \Leftrightarrow P \wedge \neg Q$
4.	$(\exists y)\neg(M(y) \rightarrow W(y))$	EG,(3)	
5.	$\neg(y)(M(y) \rightarrow W(y))$	T	$(A)\ \neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
6.	$(\exists x)(F(x) \wedge S(x)) \rightarrow (y)(M(y) \rightarrow W(y))$	P,	Given.
7.	$\neg \exists(x)(F(x) \wedge S(x))$	T	(5),(6) $\neg Q, P \rightarrow Q \Rightarrow \neg P$ Modus tollens.
8.	$(x)\neg(F(x) \wedge S(x))$	T	$\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$
9.	$\neg(F(x) \wedge S(x))$	US, (8)	
10.	$F(x) \rightarrow \neg S(x)$	T	$\neg(P \wedge Q) \Leftrightarrow \neg P \vee \neg Q$ $(P \rightarrow Q) \Leftrightarrow \neg P \vee Q$.
11.	$(x)(F(x) \rightarrow \neg S(x))$	UG,(10)	

6. Using CP (or) otherwise obtain the following implication

$(\forall x)(P(x) \to Q(x)),\ (\forall x)(R(x) \to \neg Q(x)) \Rightarrow (\forall x)(R(x) \to \neg P(x))$

<u>Solu:</u>

Steps	derivation	rule	reason
1.	$(\forall x)(P(x) \to Q(x))$	P	Given.
2.	$(\forall x)(R(x) \to \neg Q(x))$	P	Given.
3.	$R(x) \to \neg Q(x)$	US	(2)
4.	$R(x)$	P	additional
5.	$\neg Q(x)$	T	(3),(4) $P, P \to Q \Rightarrow Q$ modus ponens
6.	$P(x) \to Q(x)$	US	(1)
7.	$\neg P(x)$	T	(5),(6) $\neg Q, P \to Q \Rightarrow \neg P$ modus tollens.
8.	$R(x) \to \neg P(x)$	CP	(4), (7)
9.	$(\forall x)(R(x) \to \neg P(x))$	UG,	(9)

7. Show that
$(x)(P(x) \vee Q(x)) \Rightarrow (x)P(x) \vee (\exists x)Q(x)$

Soln:
We shall use the indirect method of proof by assuming $\neg((x)P(x) \vee (\exists x)Q(x))$ as an additional premise.

Steps	derivation	rule	reason.
1.	$\neg[(x)P(x) \vee (\exists x)Q(x)]$	P	additional premise
2.	$\neg(x)P(x) \wedge \neg(\exists x)Q(x)$	T	(1), $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
3.	$\neg(x)P(x)$	T	(2), $P \wedge Q \Rightarrow P$
4.	$(\exists x)\neg P(x)$	T	(3), $\neg(x)A(x) \Leftrightarrow (\exists x)\neg A(x)$
5.	$\neg(\exists x)Q(x)$	T	(2), $P \wedge Q \Rightarrow Q$.
6.	$(x)\neg Q(x)$	T	(5), $\neg(\exists x)A(x) \Leftrightarrow (x)\neg A(x)$
7.	$\neg P(y)$	ES	(4)
8.	$\neg Q(y)$	US	(5)
9.	$\neg P(y) \wedge \neg Q(y)$	T	(7) (8) $P, Q \Rightarrow P \wedge Q$.
10.	$\neg[P(y) \vee Q(y)]$	T	(9), $\neg(P \vee Q) \Leftrightarrow \neg P \wedge \neg Q$
11.	$(x)(P(x) \vee Q(x))$	P	Given.
12.	$P(y) \vee Q(y)$	US.	(11)
13.	$\neg(P(y) \vee Q(y)) \wedge (P(y) \vee Q(y))$	T	(10), (12), Contradiction.