Consistency of premises and Indirect Method of proof.

Consistent:

A set of formulas $H_1, H_2 \ldots H_m$ is said to be consistent if their conjunction has the truth value T.

(ie) $(H_1 \wedge H_2 \wedge \ldots \wedge H_m) \Rightarrow A \vee \neg A$ where A is any formula.

Inconsistent:

A set of formulas $H_1, H_2 \ldots H_m$ is inconsistent if their conjunction implies a contradiction.

(ie) $(H_1 \wedge H_2 \wedge \ldots \wedge H_m) \Rightarrow R \wedge \neg R$ where R is any formula

Indirect method of Proof:

The notion of inconsistency is used in a procedure called proof by contradiction (or) reduction (or) Indirect method of proof.

Working Rule - Indirect method

1. Introduce the negation of the desired conclusion as a new premise.
2. From the new premise, together with the given premises, derive a contradiction.
3. Assert the desired conclusion as a logical inference from the premises.

1) Show that the following sets of premises are inconsistent, $P \to Q$, $P \to R$, $Q \to \neg R$, P.

Solu:

Steps	derivation	Rule	reason.
1	$P \to Q$	P	Given premise
2.	$Q \to \neg R$	P	Given Premise
3.	$P \to \neg R$	T	(1),(2) Hyp. Syllogism
4.	P	P	Given
5.	$\neg R$	T	(3),(4) Modus ponens.
6.	$P \to R$	P	Given premise
7.	$\neg P$	T	(5) (6) Modus tollens.
8	$P \wedge \neg P$	T	[(4),(7) $P, Q \Rightarrow P \wedge Q$].
9.	F		Contradiction.

2) Show that the following premises are inconsistent.
(1) If Jack misses many classes through illness, then he fails high school.
(2) If Jack fails high school, then he is uneducated.
(3) If Jack reads a lot of books, then he is not uneducated.
(4) Jack misses many classes through illness and reads a lot of books.

Soln:
E : Jack misses many classes through illness
S : Jack fails high school.
A : Jack reads a lot of books.
H : Jack is uneducated.

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The premises are $E \to S$, $S \to H$, $A \to \neg H$ & $E \wedge A$

Steps	derivation	Rule	Reason
1	$E \to S$	P	Given premise
2	$S \to H$	P	"
3	$E \to H$	T	(1),(2) Hypothetical Syllogism $P \to Q, Q \to R \Rightarrow P \to R$
4	$A \to \neg H$	P	Given Premise
5	$H \to \neg A$	T	(4), $P \to Q \Leftrightarrow \neg Q \to \neg P$
6	$E \to \neg A$	T	(3),(5) Hypo. syllo.
7	$\neg E \vee \neg A$	T	(6), $P \to Q \Leftrightarrow \neg P \vee Q$
8	$\neg(E \wedge A)$	T	(7), $\neg E \vee \neg A \Leftrightarrow \neg(E \wedge A)$
9	$E \wedge A$	P	Given premise
10.	$(E \wedge A) \wedge \neg(E \wedge A)$	T	(8),(9), $P, Q \Rightarrow P \wedge Q$
11.	F		(10), Contradiction.

By indirect proof, show that $P \to Q, Q \to R, P \vee R \Rightarrow R$

Soln:

The desired result is R. Include $\neg R$ as a new premise.

Steps	derivation	Rule	Reason
1	$Q \to R$	P	Given
2	$\neg R$	P	additional premise
3	$\neg Q$	T	(1),(2) Modus tollens.
4.	$P \to Q$	P	Given Premise
5.	$\neg P$	T	(3),(4) Modus tollens.
6.	$P \vee R$	P	Given
7	R	T	$\neg P, P \vee Q \Rightarrow Q$ dis. syllo (5),(6)
8.	$R \wedge \neg R$	T	(2),(7) contradiction.

(4) Show that $\neg(P \wedge Q)$ follows from $\neg P \wedge \neg Q$.

Solu. We introduce $\neg\neg(P \wedge Q)$ as an additional premise and show that this additional premise leads to contradiction

Steps	derivation	Rule	Reason
1	$\neg\neg(P \wedge Q)$	P	additional premise
2	$P \wedge Q$	T	(1), $\neg\neg P \Leftrightarrow P$
3	P	T	(2), $P \wedge Q \Rightarrow P$
4	$\neg P \wedge \neg Q$	P	Given premise
5.	$\neg P$	T	(4), $P \wedge Q \Rightarrow P$
6.	$P \wedge \neg P$	T	(3),(5) $P, Q \Rightarrow P \wedge Q$
7.	F		(6) Contradiction,

(5) Show that the following implication by using indirect method. $(R \rightarrow \neg Q), R \vee S, S \rightarrow \neg Q, P \rightarrow Q \Rightarrow \neg P$.

Solu:

To use the indirect method, we will include $\neg\neg P \Leftrightarrow P$ as an additional premise and prove a contradiction.

Steps	derivation	Rule	Reason
1	P	P	additional premise.
2	$P \rightarrow Q$	P	Given premise
3	Q	T	(1) (2) Modus ponens
4	$R \rightarrow \neg Q$	P	Given premise
5	$S \rightarrow \neg Q$	P	Given premise
6.	$(R \vee S) \rightarrow \neg Q$	T	[(4), (5) equivalence. $P \rightarrow R, Q \rightarrow R \Rightarrow P \vee Q \rightarrow R$.]
7.	$R \vee S$	P	Given premise.
8.	$\neg Q$	T	(6), (7) Modus ponens.
9.	$Q \wedge \neg Q$	T	(3), (8) $P, Q \Rightarrow P \wedge Q$
10.	F		(9). Contradiction.