1

UNIT – V

COMPLEX INTEGRATION

Cauchy's Integral Theorem :

If $f(z)$ is analytic and $f'(z)$ is continuous on and inside a simple closed curve C, then $\int_C f(z)\,dz = 0$

Problems :

1) Evaluate $\int_C \frac{dz}{z+4}$, where C is the circle $|z| = 2$.

Soln :

$z+4 = 0$

$z = -4$

C is $|z| = 2$

$z = -4 \Rightarrow |z| = |-4| = 4 > 2$

$\therefore z = -4$ lies outside C.

By Cauchy's Integral theorem, $\int_C \frac{dz}{z+4} = 0$.

2) Evaluate $\int_C \frac{dz}{2z-3}$, where C is the circle $|z| = 1$.

Soln :

$2z-3 = 0 \Rightarrow z = 3/2$

C is $|z| = 1$

$z = 3/2 \Rightarrow |z| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$

$z = \frac{3}{2}$ lies outside C.

By Cauchy's Integral theorem, $\int_C \frac{dz}{2z-3} = 0$.

3) Evaluate $\int_C e^z\,dz$ where C is $|z| = 1$.

Soln :

$f(z) = e^z$

$\therefore f(z)$ lies inside C.

$\therefore \int_C e^z\,dz = 0$ [By C.I.T]

Cauchy's Integral formula :

Let $f(z)$ be an analytic function inside and on a simple closed contour c, taken in the positive sense. If a is any point interior to c, then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{z-a}$$

(or).

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i\, f(a).$$

Note :

$$\int_c \frac{f(z)}{z-a}\,dz = \begin{cases} 2\pi i\, f(a), & a \text{ lies inside } c \\ 0, & a \text{ lies outside } c \end{cases}$$

$$\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz = \begin{cases} \frac{2\pi i}{n!} f^n(a), & a \text{ lies inside } c \\ 0, & a \text{ lies outside } c \end{cases}$$

Problems :

1) Evaluate $\int_c \frac{z}{z-2}\,dz$ where c is $|z| = 1$ & $|z| = 3$.

Soln :

Given $\int_c \frac{z}{z-2}\,dz$

$z - 2 = 0 \Rightarrow z = 2$

$\therefore z$ lies outside c, $|z| = 2$

$$\therefore \int_c \frac{z}{z-2}\,dz = 0.$$

z lies outside c, $|z| = 3$.

$$\int_c \frac{z}{z-2}\,dz = 2\pi i\, f(2)$$
$$= 2\pi i\,(2) = 4\pi i$$

2) Evaluate $\int_c \frac{z+1}{z^3+2z+4}\,dz$ where c is the circle $|z+1+i| = 2$ using cauchy's integral formula.

Soln:

Given, $\int_C \frac{z+1}{z^2+2z+4}\,dz$

$z^2+2z+4=0$

$z = \frac{-2\pm\sqrt{4-16}}{2} = \frac{-2\pm\sqrt{-12}}{2} = \frac{-2\pm 2i\sqrt{3}}{2} = -1\pm\sqrt{3}i$

$$\int \frac{z+1}{[z-(-1+\sqrt{3}i)]\,[z-(-1-\sqrt{3}i)]}\,dz$$

$|z+1+i| = |-1+\sqrt{3}i+1+i| = |(\sqrt{3}+1)i|$

$= 2.732 > 2$

$z=-1+\sqrt{3}i$ lies outside C.

$|z+1+i| = |-1-\sqrt{3}i+1+i| = |(1-\sqrt{3})i|$.

$= |-1.732+1| = -0.732 < 2$

$\therefore z = -1-\sqrt{3}i$ lies inside C.

$$\int_C \frac{z+1}{z^2+2z+4}\,dz = \int \frac{z+1}{[z-(-1+\sqrt{3}i)]\,[z-(-1-\sqrt{3}i)]}\,dz$$

$$= \int \frac{\frac{z+1}{z-(-1+\sqrt{3}i)}}{z-(-1-\sqrt{3}i)}\,dz$$

$$= 2\pi i\, f(-1-\sqrt{3}i)$$

$$= 2\pi i\left[\frac{-1-\sqrt{3}i+1}{-1-\sqrt{3}i+1-\sqrt{3}i}\right]$$

$$= 2\pi i\left[\frac{-\sqrt{3}i}{-2\sqrt{3}i}\right] = \pi i$$

3) Evaluate $\int_C \frac{e^z}{z-1}\,dz$ if C is $|z|=2$.

Soln:

Given, $\int_C \frac{e^z}{z-1}\,dz$

$z-1=0 \Rightarrow z=1$

$\therefore z=1$ lies inside C, $|z|=2$

$$\int_C \frac{e^z}{z-1}\,dz = 2\pi i\, f(1) = 2\pi i\, e^1 = 2\pi i e.$$

4) Evaluate $\int_C \frac{z+4}{z^2+2z+5}\,dz$ where c is the circle $|z+1+i|=2$ using cauchy's integral formula.

Soln :

Given, $\int_C \frac{z+4}{z^2+2z+5}\,dz$

$z^2+2z+5=0$

$$z = \frac{-2 \pm \sqrt{4-20}}{2} = \frac{-2 \pm 4i}{2} = -1 \pm 2i$$

$|z+1+i| = |-1+2i+1+i| = |3i| = 3 > 2$

$\therefore z = -1+2i$ lies outside c.

$|z+1+i| = |-1-2i+1+i| = |-i| = 1 < 2$

$\therefore z = -1-2i$ lies inside c.

$$\int_C \frac{z+4}{z^2+2z+5}\,dz = \int_C \frac{z+4}{[z-(-1+2i)][z-(-1-2i)]}\,dz$$

$$= \int_C \frac{\frac{z+4}{z-(-1+2i)}}{z-(-1-2i)}\,dz$$

$$= 2\pi i\, f(-1-2i)$$

$$= 2\pi i \left[\frac{-1-2i+4}{-1-2i+1-2i}\right]$$

$$= 2\pi i \left[\frac{3-2i}{-4i}\right] = -\frac{(3-2i)}{2}\pi$$

$$= \left(\frac{-3+2i}{2}\right)\pi$$

5

5) Find $\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz$ where C is $|z| = 3$.

Soln:

Given, $f(z) = \int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)}$

$(z-1)(z-2) = 0 \Rightarrow z = 1, 2 < 3$ lies ~~outside~~ inside C, $|z| = 3$.

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1$, $\Rightarrow A = -1$

Put $z = 2$, $\Rightarrow B = 1$

$$\frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} = -\frac{(\sin \pi z^2 + \cos \pi z^2)}{z-1} + \frac{\sin \pi z^2 + \cos \pi z^2}{z-2}$$

$$\int_C \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -\int \frac{\sin \pi z^2 + \cos \pi z^2}{z-1} dz + \int \frac{\sin \pi z^2 + \cos \pi z^2}{z-2} dz$$

$$= -2\pi i f(1) + 2\pi i f(2)$$

$$= -2\pi i(-1) + 2\pi i = 4\pi i.$$

6) Evaluate $\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz$ by cauchy's integral formula, where C is the circle $|z| = 1$.

Soln:

$$\left(z - \frac{\pi}{6}\right)^3 = 0 \Rightarrow z = \frac{\pi}{6}$$

$$|z| = \left|\frac{\pi}{6}\right| = 0.5233 < 1.$$

$z = \pi/6$ lies inside C, $|z| = 1$.

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3} dz = \frac{2\pi i}{2!} \frac{d^2}{dz^2} [\sin^6 z]_{z=\pi/6}$$

$$\frac{d^2}{dz^2}[\sin^6 z] = \frac{d}{dz}\left[\frac{d}{dz}\sin^6 z\right]$$

$$= \frac{d}{dz}\left[6\sin^5 z \cos z\right]$$

$$= 6\left[\sin^5 z(-\sin z) + \cos z\,(5\sin^4 z\cos z)\right]$$

$$= 6\left[-\sin^6 z + 5\sin^4 z\cos^2 z\right]$$

$$\int_C \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = \frac{2\pi i \times 6}{2!}\left[-\sin^6 z + 5\sin^4 z\cos^2 z\right]_{z=\pi/6}$$

$$= 6\pi i\left[-\sin^6\frac{\pi}{6} + 5\sin^4\left(\frac{\pi}{6}\right)\cos^2\left(\frac{\pi}{6}\right)\right]$$

$$= 6\pi i\left[-\frac{1}{64} + 5\times\frac{3}{4}\times\frac{1}{16}\right]$$

$$= 6\pi i\left[\frac{-1+15}{64}\right] = \frac{21\pi i}{16}$$

7) Using cauchy's integral formula evaluate $\int_C \frac{e^z}{(z+2)(z+1)^2}dz$ where C is $|z| = 3$.

Soln:

$(z+2)(z+1)^2 = 0$

$z = -2,\ z = -1$

$|z| = |-2| = 2 < 3$

$\therefore z = -2$ lies inside C, $|z| = 3$.

$|z| = |-1| = 1 < 3$

$\therefore z = -1$ lies inside C, $|z| = 3$.

$$\int_C \frac{e^z}{(z+2)(z+1)^2}\,dz = \int_C \frac{\frac{e^z}{(z+1)^2}}{z+2}\,dz + \int_C \frac{\frac{e^z}{z+2}}{(z+1)^2}\,dz$$

$$= 2\pi i\, f(-2) + 2\pi i\, f'(-1)$$

$$= 2\pi i\,\frac{e^{-2}}{(-1)^2} + 2\pi i\left[\frac{(z+2)e^z - e^z(1)}{(z+2)^2}\right]_{z=-1}$$

$$= 2\pi i\, e^{-2} + 2\pi i\left[\frac{(1)e^{-1} - e^{-1}}{(-1+2)^2}\right]$$

$= 2\pi i e^{-2} + 2\pi i [e^{-1} - e^{-1}]$

$= 2\pi i e^{-2}$

8) Evaluate $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$ where C is the circle $|z| = 3/2$.

Soln:

Given, $\int_C \frac{4-3z}{z(z-1)(z-2)} dz$

$z = 0, 1, 2$

$|z| = 0 < 3/2$

$|z| = |1| = 1 < 3/2$

$z = 0, 1$ lies inside C.

$|z| = |2| = 2 > 3/2$

$\therefore z = 2$ lies outside C.

$$\int_C \frac{4-3z}{z(z-1)(z-2)} dz = \int \frac{\frac{4-3z}{z-2}}{z(z-1)} dz$$

$$\frac{1}{z(z-1)} = \frac{A}{z} + \frac{B}{z-1} = \frac{A(z-1)+B(z)}{z(z-1)}$$

$1 = A(z-1) + B(z)$

Put $z = 0 \Rightarrow A = -1$

Put $z = 1 \Rightarrow B = 1$

$$\frac{1}{z(z-1)} = \frac{-1}{z} + \frac{1}{z-1}$$

$$\int_C \frac{\frac{4-3z}{z-2}}{z(z-1)} dz = -\int \frac{\frac{4-3z}{z-2}}{z\cancel{(z-1)}} dz + \int_C \frac{\frac{4-3z}{z-2}}{z-1} dz$$

$= -2\pi i f(0) + 2\pi i f(1)$

$= -2\pi i \left(\frac{4}{-2}\right) + 2\pi i \left(\frac{1}{-1}\right) = 4\pi i - 2\pi i = 2\pi i$

9) Evaluate $\int_C \frac{z\, dz}{(z-1)(z-2)^2}$ where C is $|z-2| = \frac{1}{2}$ using Cauchy's integral formula.

Soln:

$(z-1)(z-2) = 0$

$z = 1, 2$

$|z-2| = |1-2| = |-1| = 1 > 1/2$

$\therefore z = 1$ lies outside C.

$|z-2| = |2-2| = 0 < 1/2$

$\therefore z = 2$ lies inside C.

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{\frac{z}{z-1}}{(z-2)^2}\,dz$$

$$= 2\pi i\, f'(2)$$

$$= 2\pi i \left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]_{z=2}$$

$$= 2\pi i \left[\frac{1-2}{1^2}\right] = 2\pi i(-1) = -2\pi i.$$

Taylor's Series :

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2 \frac{f''(a)}{2!} + \cdots$$

This is known as Taylor's series of $f(z)$ at $z = a$.

Maclaurin's Series :

Put $a = 0$ in the Taylor series for $f(z)$ then

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

This series is called Maclaurin's series of $f(z)$.

Problems :

1) Expand $f(z) = \sin z$ in a Taylor series about $z = 0$.

Soln :

Function	At $z = 0$
$f(z) = \sin z$	$f(0) = 0$
$f'(z) = \cos z$	$f'(0) = 1$
$f''(z) = -\sin z$	$f''(0) = 0$
$f'''(z) = -\cos z$	$f'''(0) = -1$
$f^{iv}(z) = \sin z$	$f^{iv}(0) = 0$