



$$(4) \Gamma_{n+1} = \int_0^{\infty} x^n e^{-x} dx$$

$$(5) \Gamma_{n+1} = n \Gamma_n$$

$$(6) \Gamma_{1/2} = \sqrt{\pi}$$

$$(7) \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$(8) \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$(9) \sin^3 \theta = \frac{1}{4} [3 \sin \theta - \sin 3 \theta]$$

$$(10) \cos^3 \theta = \frac{1}{4} [\cos 3 \theta + 3 \cos \theta]$$

$$(11) \sin A \cos B = \frac{1}{2} [\sin (A + B) + \sin (A - B)]$$

$$(12) \cos A \sin B = \frac{1}{2} [\sin (A + B) - \sin (A - B)]$$

$$(13) \cos A \cos B = \frac{1}{2} [\cos (A + B) + \cos (A - B)]$$

$$(14) \sin A \sin B = -\frac{1}{2} [\cos (A + B) - \cos (A - B)]$$

5.2 TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Result (1) : Prove that $L[1] = \frac{1}{s}$ where $s > 0$

Proof : We know that $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

Here $f(t) = 1$

$$\begin{aligned} \therefore L[1] &= \int_0^{\infty} e^{-st} dt = \left[\frac{e^{-st}}{-s} \right]_0^{\infty} \\ &= -\frac{1}{s} \left[e^{-st} \right]_0^{\infty} = -\frac{1}{s} [e^{-\infty} - e^{-0}] \\ &= -\frac{1}{s} [0 - 1] \text{ by note (2)} \\ &= \frac{1}{s}, s > 0 \end{aligned}$$



Result (2) : Prove that $L [t^n] = \frac{n!}{s^{n+1}}$ [$n = 0, 1, 2, \dots$]

Proof : We know that

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$\begin{aligned} L[t^n] &= \int_0^{\infty} e^{-st} t^n dt = \int_0^{\infty} t^n d \left[\frac{e^{-st}}{-s} \right] \\ &= t^n \left[\frac{e^{-st}}{-s} \right]_0^{\infty} - \int_0^{\infty} \frac{e^{-st}}{-s} n t^{n-1} dt \\ &= (0 - 0) + \frac{n}{s} \int_0^{\infty} e^{-st} t^{n-1} dt \end{aligned}$$

i.e., $L[t^n] = \frac{n}{s} L[t^{n-1}]$

Similarly $L[t^{n-1}] = \frac{n-1}{s} L[t^{n-2}]$

$$L[t^{n-2}] = \frac{n-2}{s} L[t^{n-3}]$$

.....

$$\begin{aligned} L[t^{n-(n-1)}] &= \frac{n - (n - 1)}{s} L[t^{n - (n - 1) - 1}] \\ &= \frac{1}{s} L[t^0] = \frac{1}{s} L[1] = \frac{1}{s} \frac{1}{s} \end{aligned}$$

$$\begin{aligned} \therefore L[t^n] &= \frac{n}{s} \frac{n-1}{s} \dots \frac{2}{s} \frac{1}{s} \frac{1}{s} = \frac{n!}{s^n} \frac{1}{s} \\ &= \frac{n!}{s^{n+1}} \text{ where } [n = 0, 1, 2, \dots] \end{aligned}$$

Result (3) Prove that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$ where n is not a integer.

Proof : We know that

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$

$$L[t^n] = \int_0^{\infty} e^{-st} t^n dt$$



$$\text{Put } st = x \quad \text{as } t \rightarrow 0 \Rightarrow x \rightarrow 0$$

$$s \, dt = dx \quad \text{as } t \rightarrow \infty \Rightarrow x \rightarrow \infty$$

$$= \int_0^{\infty} e^{-x} \left(\frac{x}{s}\right)^n \frac{dx}{s}$$

$$= \int_0^{\infty} e^{-x} \frac{x^n}{s^{n+1}} dx$$

$$= \frac{1}{s^{n+1}} \int_0^{\infty} x^n e^{-x} dx$$

$$\text{i.e., } L\{t^n\} = \frac{\Gamma_{n+1}}{s^{n+1}} \quad \left[\because \int_0^{\infty} x^n e^{-x} dx = \Gamma_{n+1} \right]$$

when n is a positive integer.

we get $\Gamma_{n+1} = n!$

$$L\{t^n\} = \frac{n!}{s^{n+1}}$$

II. PROBLEMS BASED ON TRANSFORMS OF ELEMENTARY FUNCTIONS - BASIC PROPERTIES

Example 1 Find $L\{t\}$

Solution : $L\{t^n\} = \frac{n!}{s^{n+1}}$ [we know that]

$$L\{t\} = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$$

Example 2 Find $L\{t^3\}$

Solution : We know that $L\{t^n\} = \frac{n!}{s^{n+1}}$

$$L\{t^3\} = \frac{3!}{s^{3+1}} = \frac{6}{s^4}$$

Example 3 Find $L\{\sqrt{t}\}$

Solution : We know that $L\{t^n\} = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\{\sqrt{t}\} = L\{t^{1/2}\} = \frac{\Gamma_{1/2+1}}{s^{1/2+1}}$$



$$= \frac{\frac{1}{2} \Gamma_{1/2}}{s^{3/2}} \quad [\because \Gamma_{n+1} = n \Gamma_n ; \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{\Gamma_{1/2}}{2s^{3/2}} = \frac{\sqrt{\pi}}{2s^{3/2}}$$

Example 4. Find L [t^{3/2}]

Solution :

We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L[t^{3/2}] = \frac{\Gamma_{3/2+1}}{s^{3/2+1}} = \frac{\frac{3}{2} \Gamma_{3/2}}{s^{5/2}}$$

$$= \frac{\frac{3}{2} \Gamma_{1/2+1}}{s^{5/2}} = \frac{\left(\frac{3}{2}\right) \left(\frac{1}{2}\right) \Gamma_{1/2}}{s^{5/2}}$$

$$= \frac{\left(\frac{3}{4}\right) \sqrt{\pi}}{s^{5/2}} \quad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

$$= \frac{3\sqrt{\pi}}{4s^{5/2}}$$

Example 5.2.5. Find L $\left[\frac{1}{\sqrt{t}}\right]$

Solution : We know that $L[t^n] = \frac{\Gamma_{n+1}}{s^{n+1}}$

$$L\left[\frac{1}{\sqrt{t}}\right] = L[t^{-1/2}] = \frac{\Gamma_{-1/2+1}}{s^{-1/2+1}}$$

$$= \frac{\Gamma_{1/2}}{s^{1/2}}$$

$$= \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}} \quad [\because \Gamma_{1/2} = \sqrt{\pi}]$$

Result 4. Prove that $L[e^{at}] = \frac{1}{s-a}$ where $s > a$.

Proof : We know that

$$L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$$



$$\begin{aligned}L[e^{at}] &= \int_0^{\infty} e^{-st} e^{at} dt = \int_0^{\infty} e^{-(s-a)t} dt \\&= \left[\frac{e^{-(s-a)t}}{-(s-a)} \right]_0^{\infty} = -\frac{1}{s-a} \left[e^{-(s-a)t} \right]_0^{\infty} \\&= \frac{-1}{s-a} [0 - 1] = \frac{1}{s-a} \text{ where } s-a > 0\end{aligned}$$

Example 6. Find the value $L[e^{3t}]$

Solution : We know that

$$L[e^{at}] = \frac{1}{s-a}$$

$$L[e^{3t}] = \frac{1}{s-3}$$

Example 7 Find $L[e^{3t+5}]$

Solution :

W.K.T $L[e^{at}] = \frac{1}{s-a}$

$$\begin{aligned}L[e^{3t+5}] &= L[e^{3t} e^5] \\&= e^5 L[e^{3t}] = e^5 \left[\frac{1}{s-3} \right] = \frac{e^5}{s-3}\end{aligned}$$

Example 8 Find $L\left[\frac{e^{at}}{a}\right]$

Solution : W.K.T $L[e^{at}] = \frac{1}{s-a}$

$$L\left[\frac{e^{at}}{a}\right] = \frac{1}{a} L[e^{at}] = \frac{1}{a} \left[\frac{1}{s-a} \right]$$

Example 9 Find $L[2^t]$

w.K.T. $L[e^{at}] = \frac{1}{s-a}$

$$\begin{aligned}L[2^t] &= L[e^{\log 2^t}] \\&= L[e^{t \log 2}] \\&= L[e^{(\log 2)t}] \\&= \frac{1}{s - \log 2}\end{aligned}$$



Result 5. Prove that $L[e^{-at}] = \frac{1}{s+a}$, $(s+a) > 0$

Proof : W.K.T. $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$\begin{aligned} L[e^{-at}] &= \int_0^{\infty} e^{-st} e^{-at} dt \\ &= \int_0^{\infty} e^{-(s+a)t} dt \\ &= \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^{\infty} = -\frac{1}{s+a} \left[e^{-(s+a)t} \right]_0^{\infty} \\ &= -\frac{1}{s+a} [0 - 1] \\ &= \frac{1}{s+a} \text{ where } (s+a) > 0 \end{aligned}$$

Example 10. Find $L[e^{-bt}]$

Solution : W.K.T $L[e^{-at}] = \frac{1}{s+a}$

$$L[e^{-bt}] = \frac{1}{s+b}$$

Example 11. Find $L[2e^{-3t}]$

Solution : W.K.T. $L[e^{-at}] = \frac{1}{s+a}$

$$\begin{aligned} L[2e^{-3t}] &= 2L[e^{-3t}] \\ &= 2 \left[\frac{1}{s+3} \right] = \left[\frac{2}{s+3} \right] \end{aligned}$$

Result 6. Prove that $L[\sin at] = \frac{a}{s^2 + a^2}$ ($s > 0$)

Proof : W.K.T. $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$L[\sin at] = \int_0^{\infty} e^{-st} \sin at dt$$

$$\int e^{ax} b(x) dx$$

$$= \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$= \left[\frac{e^{-st}}{s^2 + a^2} [-s \sin at - a \cos at] \right]_0^{\infty} \text{ by Note 7.}$$



$$= 0 - \left[\frac{(-a)}{s^2 + a^2} \right] = \frac{a}{s^2 + a^2} \text{ where } s > 0.$$

Example 5.2.12. Find L [sin 2t]

Solution : W.K.T $L[\sin at] = \frac{a}{s^2 + a^2}$

$$\begin{aligned} L[\sin 2t] &= \frac{2}{s^2 + 2^2} \\ &= \frac{2}{s^2 + 4} \end{aligned}$$

Example 5.2.13. Find L [sin πt]

Solution : W.K.T $L[\sin at] = \frac{a}{s^2 + a^2}$

$$L[\sin \pi t] = \frac{\pi}{s^2 + \pi^2}$$

Result : 7. Prove that $L[\cos at] = \frac{s}{s^2 + a^2}$ ($s > 0$)

Proof : W.K.T. $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$\begin{aligned} L[\cos at] &= \int_0^{\infty} e^{-st} \cos at dt \\ &= \left[\frac{e^{-st}}{s^2 + a^2} [-s \cos at + a \sin at] \right]_0^{\infty} \\ &= 0 - \left[\frac{1}{s^2 + a^2} (-s) \right] \\ &= \frac{s}{s^2 + a^2} \quad (s > 0) \end{aligned}$$

Example 5.2.14. Find L [cos 2t]

Solution : W.K.T. $L[\cos at] = \frac{s}{s^2 + a^2}$

$$L[\cos 2t] = \frac{s}{s^2 + 4}$$



Example 15 Prove that $L[\cos at] = \frac{s}{s^2 + a^2}$ and $L[\sin at] = \frac{a}{s^2 + a^2}$

Solution : By Euler's theorem

$$e^{ix} = \cos x + i \sin x$$

$$e^{iat} = \cos at + i \sin at$$

$$\begin{aligned} L[e^{iat}] &= L[\cos at + i \sin at] \\ &= L[\cos at] + i L[\sin at] \end{aligned}$$

$$\begin{aligned} L[\cos at] + i L[\sin at] &= L[e^{iat}] \\ &= \frac{1}{s - ia} \\ &= \left[\frac{1}{s - ia} \right] \left[\frac{s + ia}{s + ia} \right] \\ &= \frac{s + ia}{s^2 + a^2} \end{aligned}$$

Equating real & Imaginary parts we get

$$L[\cos at] = \frac{s}{s^2 + a^2}$$

$$L[\sin at] = \frac{a}{s^2 + a^2}$$

Example 16 Find $L[\cos (at + b)]$

$$\begin{aligned} \text{Solution : } L[\cos (at + b)] &= L[\cos at \cos b - \sin at \sin b] \\ &= \cos b L[\cos at] - \sin b L[\sin at] \\ &= \cos b \left[\frac{s}{s^2 + a^2} \right] - \sin b \left[\frac{a}{s^2 + a^2} \right] \\ &= \frac{s \cos b - a \sin b}{s^2 + a^2} \end{aligned}$$

Example 17 Find $L[\sin^2 2t]$

$$\begin{aligned} \text{Solution : } L[\sin^2 2t] &= L \left[\frac{1 - \cos 4t}{2} \right] = \frac{1}{2} L[1 - \cos 4t] \\ &= \frac{1}{2} [L[1] - L[\cos 4t]] \\ &= \frac{1}{2} \left[\frac{1}{s} - \frac{s}{s^2 + 16} \right] \end{aligned}$$



Example 18 Find L [sin 5t cos 2t]

$$\begin{aligned}\text{Solution : } L[\sin 5t \cos 2t] &= \frac{1}{2} L[\sin 7t + \sin 3t] \text{ by Note 11.} \\ &= \frac{1}{2} \left[\frac{7}{s^2 + 49} + \frac{3}{s^2 + 9} \right]\end{aligned}$$

Example 19 Find L[(sin t – cos t)²]

$$\begin{aligned}\text{Solution : } L[(\sin t - \cos t)^2] &= L[\sin^2 t + \cos^2 t - 2 \sin t \cos t] \\ &= L[1 - \sin 2t] = L[1] - L[\sin 2t] \\ &= \frac{1}{s} - \frac{2}{s^2 + 4}\end{aligned}$$

Result 8. Prove that L[sinh at] = $\frac{a}{s^2 - a^2}$ where $s > |a|$

$$\text{Proof : } \sinh at = \frac{e^{at} - e^{-at}}{2}$$

$$\begin{aligned}L[\sinh at] &= L \left[\frac{e^{at} - e^{-at}}{2} \right] \\ &= \frac{1}{2} L[e^{at} - e^{-at}] = \frac{1}{2} [L[e^{at}] - L[e^{-at}]] \\ &= \frac{1}{2} \left[\frac{1}{s-a} - \frac{1}{s+a} \right] = \frac{1}{2} \left[\frac{s+a - s+a}{s^2 - a^2} \right] \\ &= \frac{1}{2} \left[\frac{2a}{s^2 - a^2} \right] = \frac{a}{s^2 - a^2}, \quad s > |a|\end{aligned}$$

Result 9. Prove that L[cosh at] = $\frac{s}{s^2 - a^2}$, $s > |a|$

$$\text{Proof : } \cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$\begin{aligned}L[\cosh at] &= L \left[\frac{e^{at} + e^{-at}}{2} \right] \\ &= \frac{1}{2} L[e^{at} + e^{-at}] = \frac{1}{2} [L[e^{at}] + L[e^{-at}]] \\ &= \frac{1}{2} \left[\frac{1}{s-a} + \frac{1}{s+a} \right] = \frac{1}{2} \left[\frac{s+a + s-a}{s^2 - a^2} \right] \\ &= \frac{1}{2} \left[\frac{2s}{s^2 - a^2} \right] = \frac{s}{s^2 - a^2}, \quad s > |a|\end{aligned}$$



Result 11. Prove that $L[f'(t)] = s L[f(t)] - f(0)$

Proof : W.K.T. $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$L[f'(t)] = \int_0^{\infty} e^{-st} f'(t) dt$$

$$= \int_0^{\infty} e^{-st} d[f(t)]$$

$$= \left[e^{-st} f(t) \right]_0^{\infty} - \int_0^{\infty} f(t) (-s) e^{-st} dt$$

$$= [0 - f(0)] + s \int_0^{\infty} e^{-st} f(t) dt$$

$$= -f(0) + s L[f(t)]$$

$$= s L[f(t)] - f(0)$$

Result 12. Prove that $L[f''(t)] = s^2 L[f(t)] - sf(0) - f'(0)$

Proof : W.K.T. $L[f(t)] = \int_0^{\infty} e^{-st} f(t) dt$

$$L[f''(t)] = \int_0^{\infty} e^{-st} f''(t) dt$$

$$= \int_0^{\infty} e^{-st} d[f'(t)]$$

$$= \left[e^{-st} f'(t) \right]_0^{\infty} - \int_0^{\infty} f'(t) (-s) e^{-st} dt$$



$$\begin{aligned}
 &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \\
 &= \int_0^{\infty} e^{-(s+a)t} f(t) dt \\
 &= \varphi(s+a)
 \end{aligned}$$

III. PROBLEMS BASED ON FIRST SHIFTING THEOREM AND SECOND SHIFTING THEOREM

Example Find $L [t^n e^{-at}]$

$$\begin{aligned}
 \text{Solution : } L[t^n e^{-at}] &= [L(t^n)]_{s \rightarrow (s+a)} \\
 &= \left[\frac{n!}{s^{n+1}} \right]_{s \rightarrow (s+a)} \\
 &= \frac{n!}{(s+a)^{n+1}}
 \end{aligned}$$

Example Find $L [e^{-at} \cos bt]$

$$\begin{aligned}
 \text{Solution : } L[e^{-at} \cos bt] &= [L[\cos bt]]_{s \rightarrow (s+a)} \\
 &= \left[\frac{s}{s^2 + b^2} \right]_{s \rightarrow (s+a)} \\
 &= \frac{s+a}{(s+a)^2 + b^2}
 \end{aligned}$$

Example Find $L [e^{at} \sinh bt]$

$$\begin{aligned}
 \text{Solution : } L[e^{at} \sinh bt] &= [L[\sinh bt]]_{s \rightarrow (s-a)} \\
 &= \left[\frac{b}{s^2 - b^2} \right]_{s \rightarrow (s-a)} = \frac{b}{(s-a)^2 - b^2}
 \end{aligned}$$

Example Find $L [e^t t^{-1/2}]$

$$\begin{aligned}
 \text{Solution : } L[e^t t^{-1/2}] &= [L[t^{-1/2}]]_{s \rightarrow (s-1)} \\
 &= \left[\frac{\Gamma_{-1/2+1}}{s^{-1/2+1}} \right]_{s \rightarrow (s-1)} = \left[\frac{\Gamma_{1/2}}{s^{1/2}} \right]_{s \rightarrow (s-1)} \\
 &= \left[\frac{\sqrt{\pi}}{\sqrt{s}} \right]_{s \rightarrow (s-1)} = \left[\sqrt{\frac{\pi}{s}} \right]_{s \rightarrow (s-1)} \\
 &= \sqrt{\frac{\pi}{s-1}}
 \end{aligned}$$