$= (1+i)\, e^{z} + c$

$\Rightarrow f(z) = \dfrac{F(z)}{1+i}$

$\Rightarrow f(z) = e^{z} + \dfrac{c}{1+i}$

Transformation

A complex valued function of complex variable $w = f(z)$ can be treated as a transformation of points of Z-plane into points of W-plane.

Invariant or Fixed point

The invariant (or) fixed points of the transformation $w = f(z)$ is given by solving the equation $z = f(z)$.

(1) Find the invariant points of z^2.

Sol

(2) Find the invariant points of the transformation $w = \frac{z-1}{z+1}$.

Sol

The invariant points are given by

$$z = \frac{z-1}{z+1}$$

$\Rightarrow z(z+1) = z-1$

$\Rightarrow z^2 + z = z - 1$

$\Rightarrow z^2 = -1 \Rightarrow z = \pm i$.

(3) Find the invariant points of $w = \frac{2z+6}{z+7}$.

Sol

The invariant points are given by

$$z = \frac{2z+6}{z+7}$$

$\Rightarrow z^2 + 7z = 2z + 6$

(4) Find the fixed points of the mapping $\omega = \frac{6z-9}{z}$

Sol

The fixed points are given by

$$z = \frac{6z-9}{z} \Rightarrow z^2 = 6z-9$$

$$\Rightarrow z^2 - 6z + 9 = 0.$$

$$\Rightarrow (z-3)^2 = 0. \Rightarrow z = 3, 3.$$

Bilinear Transformation

Any transformation of the form $\omega = \frac{az+b}{cz+d}$, $ad-bc \neq 0$ is called Bilinear transformation.

Cross - Ratio

$$(z, z_1, z_2, z_3) = \frac{(z-z_1)(z_2-z_3)}{\ldots}$$

(1) Find the Bilinear transformation which maps the point $\infty, i, 0$ of z-plane into $0, i, \infty$ of the W-plane.

Sol

Z-plane	W-plane.
$z_1 = \infty$	$w_1 = 0$
$z_2 = i$	$w_2 = i$
$z_3 = 0$	$w_3 = \infty$.

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-w_1)\times(w_2-w_3)}{(w-w_3)\times(w_2-w_1)} = \frac{(z-z_1)\times(z_2-z_3)}{(z-z_3)\times(z_2-z_1)}$$

$$\Rightarrow \frac{w-0}{i-0} = \frac{i-0}{z-0}$$

$$\Rightarrow \frac{w}{i} = \frac{i}{z} \Rightarrow w = \frac{i^2}{z}$$

$$\Rightarrow \boxed{w = \frac{-1}{}}$$

z-plane	W-plane
$z_1 = 1$	$w_1 = 0$
$z_2 = i$	$w_2 = 1$
$z_3 = -1$	$w_3 = \infty$

The Bilinear transformation is given by

$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-0}{1-0} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow w = \frac{-i(z-1)}{(z+1)} \Rightarrow \boxed{w = \frac{i-iz}{z+1}}$$

(3) Find the Bilinear transformation that

z-plane	w-plane
$z_1 = 0$	$w_1 = i$
$z_2 = -1$	$w_2 = 0$
$z_3 = i$	$w_3 = \infty$

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{w-i}{0-i} = \frac{(z-0)(-1-i)}{(z-i)(-1-0)}$$

$$\Rightarrow \frac{w-i}{-i} = \frac{z(1+i)}{(z-i)}$$

$$\Rightarrow w-i = \frac{z}{z-i} \times \left[-i(1+i)\right]$$

$$\Rightarrow w-i = \frac{z(1-i)}{z-i}$$

$$\Rightarrow w = \frac{z(1-i)}{z-i} + i = \frac{z - iz + iz + 1}{z-i}$$

(4) Find the Bilinear transformation that maps the points $1, i, -1$ of z-plane onto $i, 0, -i$ of w-plane.

Sol

z-plane	w-plane
$z_1 = 1$	$w_1 = i$
$z_2 = i$	$w_2 = 0$
$z_3 = -1$	$w_3 = -i$

The Bilinear transformation is given by

$$(w, w_1, w_2, w_3) = (z, z_1, z_2, z_3)$$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-i)(0+i)}{(w+i)(0-i)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\Rightarrow \frac{w-i}{w+i} = +i\frac{(z-1)}{(z+1)}$$

$$\frac{w \mp i + w \pm i}{w \pm i - w \mp i} = \frac{iz \mp i + z + 1}{iz - i - z - 1}$$

$$\frac{2w}{-2i} = \frac{iz - i + z + 1}{iz - i - z - 1}$$

$$\Rightarrow \quad w = \frac{+z \mp 1 \mp iz \mp i}{iz - i - z - 1}$$

$$\Rightarrow \quad w = \frac{z(i+1) \mp (1+i)}{z(i-1) - (1+i)}$$

(b) Find the bilinear transformation that transforms the points $z = 1, i, -1$ of the z-plane into the points $w = 2, i, -2$ of the W-plane.

Sol

z-plane | W-plane

$z_1 = 1$ | $w_1 = 2$

$z_2 = i$

$$\frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} = \frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}$$

$$\frac{(w-2)(i+2)}{(w+2)(i-2)} = \frac{(z-1)(i+1)}{(z+1)(i-1)}$$

$$\frac{(w-2)}{(w+2)} \times \left(\frac{-3}{5} - \frac{4}{5}i\right) = \frac{z-1}{z+1} \times -i$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{z-1}{z+1} \times \frac{-i}{\left(\frac{-3}{5} - \frac{4}{5}i\right)}$$

$$\Rightarrow \frac{w-2}{w+2} = \left(\frac{4}{5} + \frac{3}{5}i\right)\frac{z-1}{z+1}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{(a+bi)(z-1)}{z+1}$$

$$\Rightarrow \frac{w-2}{w+2} = \frac{az-a+biz-bi}{z+1}$$

$$\therefore\ w = -2 \times \frac{\left[\frac{9}{5}z+\frac{1}{5}\right]+i(z-1)\frac{3}{5}}{\left[\frac{-1}{5}z-\frac{9}{5}\right]+i(z-1)\frac{3}{5}}$$

$$\Rightarrow\ w = -2 \times \frac{(9z+1)+i(3z-3)}{(-z-9)+i(3z-3)}$$

(6) Find the bilinear transformation which maps the points $-i, 0, i$ into the points, $-1, i, 1$ respectively. Into what curve, the y-axis is transformed under this transformation?

Sol

z-plane	W-plane
$z_1 = -i$	$w_1 = -1$
$z_2 = 0$	$w_2 = i$
$z_3 = i$	$w_3 = 1$

$$\frac{(w+1)(i-1)}{(w-1)(i+1)} = \frac{(z+i)(0-i)}{(z-i)(0+i)}$$

$$\frac{(w+1)}{(w-1)} = -\frac{(z+i)}{(z-i)} \times \frac{(i+1)}{(i-1)}$$

$$\frac{w+1}{w-1} = \frac{iz-1}{z-i}$$

$$\frac{a}{b} = \frac{c}{d} \Rightarrow \frac{a+b}{a-b} = \frac{c+d}{c-d}$$

$$\frac{w+1+w-1}{w+1-w+1} = \frac{iz-1+z-i}{iz-1-z+i}$$

$$\frac{2w}{2} = \frac{(i+1)z-(1+i)}{(i-1)z-(1-i)}$$

$$\boxed{w = \frac{(i+1)z-(1+i)}{(i-1)z-(1-i)}}$$