Inverse Laplace Transform

We obtain $f(t)$ when $F(s)$ is given, then we say that inverse Laplace transform of $F(s)$ is $f(t)$.

(1) If $L[f(t)] = F(s)$, then $L^{-1}[F(s)] = f(t)$ when L^{-1} is called the inverse Laplace transform operator.

(2) If $F_1(s)$ and $F_2(s)$ are Laplace transform of $f(t)$ and $g(t)$ respectively, then

$$L^{-1}[c_1 F_1(s) + c_2 F_2(s)] = c_1 L^{-1}[F_1(s)] + c_2 L^{-1}[F_2(s)]$$

1] $L^{-1}\left(\frac{1}{s}\right) = 1$

2] $L^{-1}\left(\frac{1}{s-a}\right) = e^{at}$

3] $L^{-1}\left[\frac{a}{s^2-a^2}\right] = \frac{1}{a}\sinh at$

4] $L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$

5] $L^{-1}\left[\frac{1}{(s-a)^2+b^2}\right] = \frac{1}{b} e^{at}\sin bt$

6] $L^{-1}\left[\frac{s-a}{(s-a)^2+b^2}\right] = e^{at}\cos bt$

7) $L^{-1}\left[\frac{1}{(s-a)^2-b^2}\right] = \frac{1}{b} e^{at}\sinh bt$

8] $L^{-1}\left[\frac{s-a}{(s-a)^2-b^2}\right] = e^{at}\cosh bt$

9] $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = \frac{1}{2a} t\sin at$

10) $L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = \frac{1}{2a}[\sin at + at\cos at]$

11) $L^{-1}\left[\frac{1}{(s^2+a^2)^2}\right] = \frac{1}{2a^3}(\sin at - at\cos at)$.

12) $L^{-1}\left[\frac{s^2-a^2}{(s^2+a^2)^2}\right] = t\cos at$

13) $L^{-1}\left[\frac{1}{s^n}\right] = \frac{t^{n-1}}{(n-1)!}$

14) $L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

15) $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

16). $L^{-1}[F(s-a)] = e^{at} f(t)$

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$$L^{-1}[C_1 F_1(s) + C_2 F_2(s)] = C_1 L^{-1}[F_1(s)] + C_2 L^{-1}[F_2(s)]$$

1] $L^{-1}\left(\frac{1}{s}\right) = 1$

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15) $L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{1}{a}\sin at$

16). $L^{-1}[F(s-a)] = e^{at} f(t)$

① Find $L^{-1}\left[\frac{1}{s-3}\right]$ formula $\left[L^{-1}\left[\frac{1}{s-a}\right] = e^{at}\right]$

$$L^{-1}\left[\frac{1}{s-3}\right] = e^{3t}$$

② Find $L^{-1}\left[\frac{2s}{s^2-16}\right] = 2L^{-1}\left[\frac{s}{s^2-4^2}\right]$

$$= 2\cosh 4t$$

Convolution Theorem

The Convolution of two functions $f(t)$ and $g(t)$ is defined as

$$f(t) * g(t) = \int_0^t f(u)\, g(t-u)\, du$$

State

If $f(t)$ and $g(t)$ are functions defined for $t \geq 0$ then $L[f(t) * g(t)] = L[f(t)] \cdot L[g(t)]$

① Using Convolution theorem find $L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right]$

Solution:

$$L^{-1}[F(s) \cdot G(s)] = L^{-1}[F(s)] * L^{-1}[G(s)]$$

$$L^{-1}\left[\frac{s}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2}\right] * L^{-1}\left[\frac{1}{s^2+a^2}\right]$$

$$= \cos at * \frac{1}{a}\sin at$$

$$= \frac{1}{a}[\cos at * \sin at]$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \sin(at-au)\cos au\,du$$

$$= 1/a \int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(at-2u)]\,du.$$

$$= \frac{1}{2a}\left[(\sin at)u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[u(\sin at) + \frac{\cos a(t-2u)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \left(\frac{\cos at}{2a}\right) - \left(0+\frac{\cos at}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

$$= \frac{1}{2a}\,t\sin at$$

(2) Using Convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\left(\frac{1}{(s+a)}\right)\left(\frac{1}{(s+b)}\right)\right]$$

$$= L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$\therefore L^{-1}[F(s)\,G(s)] = L^{-1}[F(s)] * L^{-1}(G(s))$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au}e^{-b(t-u)}\,du$$

$$= \int_0^t e^{-au}e^{-bt}e^{bu}\,du$$

$$= \frac{1}{a}\int_0^t \cos au \sin a(t-u)\,du$$

$$= \frac{1}{a}\int_0^t \sin(at-au)\cos au\,du$$

$$= \frac{1}{a}\int_0^t \frac{\sin(at-au+au)+\sin(at-au-au)}{2}\,du$$

$$= \frac{1}{2a}\int_0^t [\sin at + \sin(at-2u)]\,du.$$

$$= \frac{1}{2a}\left[(\sin at)u + \left(\frac{-\cos a(t-2u)}{-2a}\right)\right]_0^t$$

$$= \frac{1}{2a}\left[u(\sin at) + \frac{\cos a(t-2u)}{2a}\right]_0^t$$

$$= \frac{1}{2a}\left[t\sin at + \left(\frac{\cos at}{2a}\right) - \left(0+\frac{\cos at}{2a}\right)\right]$$

$$= \frac{1}{2a}\left[t\sin at + \frac{\cos at}{2a} - \frac{\cos at}{2a}\right]$$

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(2) Using Convolution theorem find $L^{-1}\left[\frac{1}{(s+a)(s+b)}\right]$

$$L^{-1}\left[\frac{1}{(s+a)(s+b)}\right] = L^{-1}\left[\left(\frac{1}{(s+a)}\right)\left(\frac{1}{(s+b)}\right)\right]$$

$$= L^{-1}\left[\frac{1}{s+a}\right] * L^{-1}\left[\frac{1}{s+b}\right]$$

$$\because L^{-1}[F(s)\,G(s)] = L^{-1}[F(s)] * L^{-1}(G(s)]$$

$$= e^{-at} * e^{-bt}$$

$$= \int_0^t e^{-au} e^{-b(t-u)}\,du$$

$$= \int_0^t e^{-au} e^{-bt} e^{bu}\,du$$

$$= e^{-bt} \int_0^t e^{-(a-b)u} du$$

$$= e^{-bt} \left[\frac{e^{-(a-b)u}}{-(a-b)} \right]_0^t$$

$$= \frac{-e^{-bt}}{a-b} \left[e^{-(a-b)u} \right]_0^t$$

$$= \frac{-e^{-bt}}{a-b} \left[e^{-(a-b)t} - 1 \right]$$

$$= \frac{-e^{-bt} + e^{-bt}}{a-b} = \frac{e^{-bt} - e^{-at}}{a-b}$$

(3) Find the inverse Laplace transform by convolution theorem $\frac{1}{s^2(s+5)}$

Soln

$$L^{-1}\left[\frac{1}{s^2(s+5)}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s+5}\right]$$

$$= t * e^{-5t}$$

$$= \int_0^t u e^{-5(t-u)} du$$

$$= \int_0^t u e^{-5t+5u} du$$

$$= e^{-5t} \int_0^t u e^{5u} du$$

$$= e^{-5t} \left[u \frac{e^{5u}}{5} - (1) \frac{e^{5u}}{25} \right]_0^t$$

$$= e^{-5t} \left[u \frac{e^{5u}}{5} - \frac{e^{5u}}{25} \right]_0^t$$

$$= e^{-5t} \left[\left(t \frac{e^{5t}}{5} - \frac{e^{5t}}{25} \right) - \left(0 - \frac{1}{25} \right) \right]$$

$$= e^{-5t} \left[t \frac{e^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25} \right]$$

$$= e^{-bt}\int_0^t e^{-(a-b)u}\,du$$

$$= e^{-bt}\left[\frac{e^{-(a-b)u}}{-(a-b)}\right]_0^t$$

$$= \frac{-e^{-bt}}{a-b}\left[e^{-(a-b)u}\right]_0^t$$

$$= \frac{-e^{-bt}}{a-b}\left[e^{-(a-b)t}-1\right]$$

$$= \frac{-e^{-bt}+e^{-bt}}{a-b} = \frac{e^{-bt}-e^{-at}}{a-b}$$

(3) Find the inverse Laplace transform by convolution theorem $\frac{1}{s^2(s+5)}$

~~Soln~~

$$L^{-1}\left[\frac{1}{s^2(s+5)}\right] = L^{-1}\left[\frac{1}{s^2}\right] * L^{-1}\left[\frac{1}{s+5}\right]$$

$$= t * e^{-5t}$$

$$= \int_0^t u e^{-5(t-u)}\,du$$

$$= \int_0^t u e^{-5t+5u}\,du$$

$$= \cancel{\int_0^t} e^{-5t}\int_0^t u e^{5u}\,du$$

$$= e^{-5t}\left[u\frac{e^{5u}}{5} - (1)\frac{e^{5u}}{25}\right]_0^t$$

$$= e^{-5t}\left[u\frac{e^{5u}}{5} - \frac{e^{5u}}{25}\right]_0^t$$

$$= e^{-5t}\left[\left(t\frac{e^{5t}}{5} - \frac{e^{5t}}{25}\right) - \left(0 - \frac{1}{25}\right)\right]$$

$$= e^{-5t}\left[t\frac{e^{5t}}{5} - \frac{e^{5t}}{25} + \frac{1}{25}\right]$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{e^{-5t}}{25}$$

$$= \frac{1}{25}\left[e^{-5t} + 5t - 1\right]$$

(7) Use Convolution theorem, to find $L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$

Soln

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s+2}{s^2+4s+13} \cdot \frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{s^2+4s+13}\right] * L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+13-4}\right] * L^{-1}\left[\frac{1}{(s+2)^2+13-4}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] * L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t} L^{-1}\left[\frac{s}{s^2+3^2}\right] * e^{-2t} L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= e^{-2t}\cos 3t * \frac{1}{3} e^{-2t}\sin 3t$$

$$= \frac{1}{3}\left[e^{-2t}\cos 3t * e^{-2t}\sin 3t\right]$$

$$= \frac{1}{3}\int_0^t e^{-2u}\cos 3u \, e^{-2(t-u)} \sin(t-u)\,du$$

$$= \frac{1}{3}\int_0^t e^{-2u}\cos 3u \, e^{-2t} e^{2u} \sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u \sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u \sin(3t-3u)\,du$$

$$= \frac{t}{5} - \frac{1}{25} + \frac{e^{-5t}}{25}$$

$$= \frac{1}{25}\left[e^{-5t} + 5t - 1\right]$$

(7) Use Convolution theorem, to find $L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right]$

Soln

$$L^{-1}\left[\frac{s+2}{(s^2+4s+13)^2}\right] = L^{-1}\left[\frac{s+2}{s^2+4s+13}\cdot\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{s^2+4s+13}\right] * L^{-1}\left[\frac{1}{s^2+4s+13}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+13-4}\right] * L^{-1}\left[\frac{1}{(s+2)^2+13-4}\right]$$

$$= L^{-1}\left[\frac{s+2}{(s+2)^2+9}\right] * L^{-1}\left[\frac{1}{(s+2)^2+9}\right]$$

$$= e^{-2t}L^{-1}\left[\frac{s}{s^2+3^2}\right] * e^{-2t}L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= e^{-2t}\cos 3t * \frac{1}{3}e^{-2t}\sin 3t$$

$$= \frac{1}{3}\left[e^{-2t}\cos 3t * e^{-2t}\sin 3t\right]$$

$$= \frac{1}{3}\int_0^t e^{-2u}\cos 3u\, e^{-2(t-u)}\sin(t-u)\,du$$

$$= \frac{1}{3}\int_0^t e^{-2u}\cos 3u\, e^{-2t}e^{2u}\sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u \sin 3(t-u)\,du$$

$$= \frac{e^{-2t}}{3}\int_0^t \cos 3u \sin(3t-3u)\,du$$

$$= \frac{e^{-2t}}{3} \int_0^t \frac{1}{2}\left[\sin 3t - \sin(-3t+6u)\right] du$$

$$= \frac{e^{-2t}}{6} \int_0^t \left[\sin 3t - \sin(6u-3t)\right] du$$

$$= \frac{e^{-2t}}{6} \sin 3t \,[u]_0^t - \frac{e^{-2t}}{6} \int_0^t \sin(6u-3t)\, du$$

$$= \frac{e^{-2t}}{6} t \sin 3t - \frac{e^{-2t}}{6}\left[\frac{-\cos(6u-3t)}{6}\right]_0^t$$

$$= \frac{e^{-2t}}{6} t \sin 3t + \frac{e^{-3t}}{36}\left[\cos(6u-3t)\right]_0^t$$

$$= \frac{e^{-2t}}{6} t \sin 3t + \frac{e^{-2t}}{36}\left[\cos 3t - \cos 3t\right].$$

$$= \frac{e^{2t}}{6} t \sin 3t$$

(5) Find $\frac{s^2}{(s^2+a^2)^2}$ using Convolution

$$L^{-1}\left[\frac{s^2}{(s^2+a^2)^2}\right] = L^{-1}\left[\frac{s}{s^2+a^2} \cdot \frac{s}{s^2+a^2}\right]$$

$$= \cos at * \cos at$$

$$= \int_0^t \cos au \cos a(t-u)\, du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au + at - au) + \cos(au - at + au)\right] du$$

$$= \frac{1}{2}\int_0^t \cos at + \cos(2au - at)\, du.$$

$$= \frac{1}{2}\left[(\cos at)\, u + \frac{\sin(2au - at)}{2a}\right]_{u=0}^{u=t}$$

$$= \frac{1}{2}\left[\left(t(\cos at) + \frac{\sin(2at - at)}{2a}\right) - \left(0 + \frac{\sin(-at)}{2a}\right)\right]$$

$$= \frac{1}{2}\left[t \cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right] = \frac{1}{2}\left[t \cos at + \frac{2 \sin at}{2a}\right]$$

$$= \frac{1}{2}\left[t \cos at + \frac{\sin at}{a}\right]$$

$$= \frac{1}{2a}\left[\sin at + at \cos at\right]$$

$$= \frac{e^{-2t}}{3}\int_0^t \frac{1}{2}\left[\sin 3t - \sin(-3t+6u)\right]du$$

$$= \frac{e^{-2t}}{6}\int_0^t \left[\sin 3t - \sin(6u-3t)\right]du$$

$$= \frac{e^{-2t}}{6}\sin 3t\,[u]_0^t - \frac{e^{-2t}}{6}\int_0^t \sin(6u-3t)\,du$$

$$= \frac{e^{-2t}}{6}t\sin 3t - \frac{e^{-2t}}{6}\left[\frac{-\cos(6u-3t)}{6}\right]_0^t$$

$$= \frac{e^{-2t}}{6}t\sin 3t + \frac{e^{-2t}}{36}\left[\cos(6u-3t)\right]_0^t$$

$$= \frac{e^{-2t}}{6}t\sin 3t + \frac{e^{-2t}}{36}\left[\cos 3t - \cos 3t\right].$$

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$$= \cos at * \cos at$$

$$= \int_0^t \cos au \cos a(t-u)\,du$$

$$= \frac{1}{2}\int_0^t \left[\cos(au+at-au) + \cos(au-at+au)\right]du$$

$$= \frac{1}{2}\int_0^t \cos at + \cos(2au-at)\,du.$$

$$= \frac{1}{2}\left[(\cos at)u + \frac{\sin(2au-at)}{2a}\right]_{u=0}^{u=t}$$

$$= \frac{1}{2}\left[\left(t(\cos at) + \frac{\sin(2at-at)}{2a}\right) - \left(0 + \frac{\sin(-at)}{2a}\right)\right]$$

$$= \frac{1}{2}\left[t\cos at + \frac{\sin at}{2a} + \frac{\sin at}{2a}\right] = \frac{1}{2}\left[t\cos at + \frac{\cancel{2}\sin at}{\cancel{2}a}\right]$$

$$= \frac{1}{2}\left(t\cos at + \frac{\sin at}{a}\right)$$

$$= \frac{1}{2a}\left[\sin at + at\cos at\right]$$