i.e., $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\,ds$

PROBLEMS BASED ON INTEGRALS OF TRANSFORM

Example 8 Find $L\left[\frac{1-e^t}{t}\right]$

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty \varphi(s)\,ds = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{1-e^t}{t}\right] = \int_s^\infty L[1-e^t]\,ds = \int_s^\infty \left[\frac{1}{s}-\frac{1}{s-1}\right]ds$$

$$= \Big[\log s - \log(s-1)\Big]_s^\infty = \left[\log\frac{s}{s-1}\right]_s^\infty$$

$$= \left[\log\frac{s}{s(1-1/s)}\right]_s^\infty = \left[\log\frac{1}{1-1/s}\right]_s^\infty$$

$$= 0 - \log\frac{s}{s-1} = \log\left(\frac{s-1}{s}\right)$$

Example 9 Find $L\left[\frac{\sin at}{t}\right]$ **[A.U., March 1996]**

Solution : $L\left[\frac{1}{t}f(t)\right] = \int_s^\infty L[f(t)]\,ds$

$$L\left[\frac{\sin at}{t}\right] = \int_s^\infty L[\sin at]\,ds = \int_s^\infty \frac{a}{s^2+a^2}\,ds$$

$$= a\left[\frac{1}{a}\tan^{-1}\left(\frac{s}{a}\right)\right]_s^\infty = \left[\tan^{-1}\frac{s}{a}\right]_s^\infty$$

$$= \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right) = \cot^{-1}\left[\frac{s}{a}\right] = \tan^{-1}\left[\frac{a}{s}\right]$$

Note : $\cot^{-1}\left(\frac{s}{a}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)$

$$\frac{s}{a} = \cot\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{s}{a}\right)\right]$$

$$= \tan\left[\tan^{-1}\left(\frac{s}{a}\right)\right] = \frac{s}{a}$$

PROBLEMS BASED ON INITIAL VALUE AND FINAL VALUE THEOREM

Example 5.4.1. If $L[f(t)] = \frac{1}{s(s+a)}$, find $\underset{t \to \infty}{Lt} f(t)$ and $\underset{t \to 0}{Lt} f(t)$

Solution : $\underset{t \to 0}{Lt} f(t) = \underset{s \to \infty}{Lt} s \cdot F(s)$

$$= \underset{s \to \infty}{Lt} s \frac{1}{s(s+a)} = \underset{s \to \infty}{Lt} \frac{1}{s+a} = \frac{1}{\infty} = 0$$

$$\underset{t \to \infty}{Lt} f(t) = \underset{s \to 0}{Lt} s F(s) = \underset{s \to 0}{Lt} s \frac{1}{s(s+a)}$$

$$= \underset{s \to 0}{Lt} \frac{1}{s+a} = \frac{1}{a}$$

Example 2. Verify the initial and final value theorem for the function $f(t) = 1 + e^{-t}(\sin t + \cos t)$

Solution : Initial value theorem states that

$$\lim_{t \to 0} f(t) = \lim_{s \to \infty} s F(s)$$

$$L[f(t)] = F(s) = \frac{1}{s} + L[\sin t + \cos t]_{s \to s+1}$$

$$= \frac{1}{s} + \frac{1}{(s+1)^2 + 1} + \frac{s+1}{(s+1)^2 + 1}$$

$$= \frac{1}{s} + \frac{s+2}{(s+1)^2 + 1}$$

$$\text{L.H.S} = \lim_{t \to 0} f(t) = 1 + 1 = 2$$

$$\text{R.H.S} = \lim_{s \to \infty} s \left[\frac{1}{s} + \frac{s+2}{(s+1)^2 + 1}\right]$$

$$= \lim_{s \to \infty} \left[1 + \frac{s(s+2)}{(s+1)^2 + 1}\right] = \lim_{s \to \infty} \left[1 + \frac{s^2\left(1 + \frac{2}{s}\right)}{s^2\left[1 + \frac{2}{s} + \frac{2}{s^2}\right]}\right]$$

$$= \lim_{s \to \infty} \left[1 + \frac{1 + \frac{2}{s}}{1 + \frac{2}{s} + \frac{2}{s^2}}\right] = 1 + 1 = 2$$

$$\text{L.H.S.} = \text{R.H.S.}$$

Initial value theorem verified.

TRANSFORM OF PERIODIC FUNCTIONS

Define periodic function and state its Laplace transform formula.

◆ Def. Periodic

A function $f(x)$ is said to be "periodic" if and only if $f(x+p) = f(x)$ is true for some value of p and every value of x. The smallest positive value of p for which this equation is true for every value of x will be called the period of the function.

The Laplace Transformation of a periodic function $f(t)$ with period p given by $\frac{1}{1-e^{-ps}} \int_0^p e^{-st} f(t)\, dt$

Proof : $L[f(t)] = \int_0^\infty e^{-st} f(t)\, dt$

$$= \int_0^p e^{-st} f(t)\, dt + \int_p^\infty e^{-st} f(t)\, dt$$

Put $t = u + p$ in the second integral

i.e., $u = t - p$ $\quad t \to p \Rightarrow u \to 0$

i.e., $du = dt$ $\quad t \to \infty \Rightarrow u \to \infty$

$$= \int_0^p e^{-st} f(t)\, dt + \int_0^\infty e^{-(u+p)s} f(u+p)\, du$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-su} f(u)\, du \quad [\because f(u+p) = f(u)]$$

$$= \int_0^p e^{-st} f(t)\, dt + e^{-sp} \int_0^\infty e^{-st} f(t)\, dt \quad [\because u \text{ is a dummy variable}]$$

$$L[f(t)] = \int_0^p e^{-st} f(t)\, dt + e^{-sp}\, L[f(t)]$$

$$[1 - e^{-sp}]\, L[f(t)] = \int_0^p e^{-st} f(t)\, dt$$

$$L[f(t)] = \frac{1}{1-e^{-sp}} \int_0^p e^{-st} f(t)\, dt$$