$$= \frac{-3}{z+1} + \frac{1}{z+1} + \left(\frac{1}{z+1}\right)^2 + \left(\frac{1}{z+1}\right)^3 + \left(\frac{1}{z+1}\right)^4 + \cdots$$

$$-\frac{2}{3}\left[1 + \frac{z+1}{3} + \left(\frac{z+1}{3}\right)^2 + \left(\frac{z+1}{3}\right)^3 + \cdots\right]$$

Singularities :

The point $z=a$ at which the function $f(z)$ is not analytic is called a singular point.

Ex:

Let $f(z) = \frac{1}{z-2}$

Here $z=2$ is a singular point.

Types of singularities

i) Isolated singularities

The point $z=a$ is said to be isolated singularity if the neighbourhood of $z=a$ contains no other singularity.

Ex:

Let $f(z) = \frac{1}{z-3}$

The function is not analytic only at $z=3$.

$z=3$ is an isolated singularities.

ii) Removable singularity

A point $z=a$ is called a removable singularities of $f(z)$ if $\lim_{z\to a} f(z) = a$, finite.

Ex:

$f(z) = \frac{\tan z}{z}$

Here $z=0$ is a singular point

$\lim_{z\to 0} \frac{\tan z}{z} = \frac{\tan 0}{0} = \frac{0}{0}$ (indefinite finite)

Applying L-Hospital rule

$\lim_{z\to 0} \frac{\tan z}{z} = \lim_{z\to 0} \frac{\sec^2 z}{1} = \sec^2 0 = \frac{1}{\cos^2 0} = 1.$

$\lim_{z\to 0} \frac{\tan z}{z} = 1$ (finite value)

The singularity $z = a$ of $f(z)$ is called a pole if there exists a +ve integer n such that

$$\lim_{z \to a} (z-a)^n f(z) \neq 0$$

If $n = 1$ it is called a simple pole.
If $n = 2$, it is called a double pole.

Ex:

$$f(z) = \frac{1}{(z-1)(z+2)^2}$$

The singularities are $1, -2$

Take $z = 1$

$$\lim_{z \to 1} (z-1) f(z) = \lim_{z \to 1} (z-1) \frac{1}{(z-1)(z+2)^2} = \frac{1}{9} \neq 0$$

$z = 1$ is a simple pole.

Take $z = -2$

$$\lim_{z \to -2} (z+2) f(z) = \lim_{z \to -2} (z+2)^2 \frac{1}{(z-1)(z+2)^2}$$

$$= -\frac{1}{3} \neq 0$$

$\therefore z = -2$ is a pole of order 2 (double pole)

Essential Singularity

A singular point $z = a$ is said to be an essential singular point of $f(z)$ if the Laurent series of $f(z)$ about $z = a$ possess of infinite number of terms in the principal part.

Ex:

$$f(z) = e^{\frac{1}{z-1}}$$

Here $z = 1$ is a singular point

At $z = 1$, $f(z) = e^{1/0} = e^{\infty}$ (which is not defined)

Also $z = 1$ is not a pole (or) removable singularity

$\therefore z = 1$ is an essential singularity

Residues :

If $z = a$ is an isolated singular point of $f(z)$ about $z = a$

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

The co-efficient of b of $\frac{1}{z-a}$ is called Residue of $f(z)$ at $z = a$.

i) If $z = a$ is a simple pole then

$$\underset{z=a}{\text{Res}}\, f(z) = \underset{z \to a}{\text{lt}} (z-a) f(z).$$

ii) If $z = a$ is a pole of order n then

$$\underset{z=a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \underset{z \to a}{\text{lt}} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

Problems under Residues :

1) Find the residue of the function $f(z) = \frac{4}{z^3(z-2)}$ at a simple pole.

Soln :

Here $z = 2$ is a simple pole.

$$\underset{z \to a}{\text{Res}}\, f(z) = \underset{z \to a}{\text{lt}} (z-a) f(z)$$

$$\underset{z \to 2}{\text{Res}}\, f(z) = \underset{z \to 2}{\text{lt}} (z-2) \frac{4}{z^3(z-2)} = \frac{4}{2^3} = \frac{1}{2}$$

2) Calculate the residue of $f(z) = \frac{e^{2z}}{(z+1)^2}$ at its pole.

Soln :

$$f(z) = \frac{e^{2z}}{(z+1)^2}$$

$z = -1$ is a pole of order 2.

$$\underset{z \to a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!} \underset{z \to a}{\text{lt}} \frac{d^{n-1}}{dz^{n-1}} \left\{ (z-a)^n f(z) \right\}$$

$$\underset{z \to -1}{\text{Res}}\, f(z) = \frac{1}{1!} \underset{z \to -1}{\text{lt}} \frac{d}{dz} \left\{ (z+1)^2 \frac{e^{2z}}{(z+1)^2} \right\}$$

$$= \underset{z \to -1}{\text{lt}} 2e^{2z} = 2e^{-2}$$

3) Find the residue of $f(z) = \dfrac{z^2}{(z-1)^2(z+2)}$ at each poles:

Soln:

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$

$z=1$ is a pole of order 2.
$z=-2$ is a pole of order 1.

$$\underset{z\to a}{\text{Res}}\, f(z) = \underset{z\to a}{\text{Lt}}\, (z-a) f(z).$$

$$\underset{z\to -2}{\text{Res}}\, f(z) = \underset{z\to -2}{\text{Lt}}\, (z+2)\frac{z^2}{(z-1)^2(z+2)} = \frac{4}{9}$$

$$\underset{z\to a}{\text{Res}}\, f(z) = \frac{1}{(n-1)!}\underset{z\to a}{\text{Lt}}\, \frac{d^n}{dz^{n-1}}(z-a)^n f(z)$$

$$\underset{z\to 1}{\text{Res}}\, f(z) = \frac{1}{1!}\underset{z\to 1}{\text{Lt}}\, \frac{d}{dz}(z-1)^2\frac{z^2}{(z-1)^2(z+2)}$$

$$= \underset{z\to 1}{\text{Lt}}\, \frac{d}{dz}\frac{z^2}{z+2}$$

$$= \underset{z\to 1}{\text{Lt}}\, \frac{(z+2)\,2z - z^2(1)}{(z+2)^2} = \frac{3(2)-1}{3^2} = \frac{5}{9}$$

Residue using Laurent series:

$[\text{Res}\, f(z)]_{z=a}$ = co-efficient of $\frac{1}{z-a}$ in the Laurent series of $f(z)$ about $z=a$.

Problems:

1) Obtain the Laurent expansion of the function $\dfrac{e^z}{(z-1)^2}$.

Soln:

$$f(z) = \frac{e^z}{(z-1)^2}$$

$z=1$ is a pole of order 2.

Put $z-1=u$.

$z = u+1$

$$f(z) = \frac{e^{u+1}}{u^2} = \frac{e^u \cdot e}{u^2}$$

$$= \frac{e}{u^2}\left[1+\frac{u}{1!}+\frac{u^2}{2!}+\frac{u^3}{3!}+\cdots\right]$$

$$= \frac{e}{u^2} + \frac{e}{1!u} + \frac{e}{2!} + \frac{e\cdot u}{3!} + \cdots$$

$$= \frac{e}{(z-1)^2} + \frac{e}{1!(z-1)} + \frac{e}{2!} + \frac{e(z-1)}{3!} + \cdots$$

This is the Laurent series expansion of $f(z)$ about $z=1$.

$$[\text{Res } f(z)]_{z=1} = \text{co-eff. of } \frac{1}{z-1} = \frac{e}{1!} = e.$$

2) Find the residues of $f(z) = \frac{z^2}{(z-1)(z+2)^2}$ at its isolated singularities using Laurent's series expansion.

<u>Soln:</u>

$$f(z) = \frac{z^2}{(z-1)(z+2)^2}$$

$z=1$ and $z=-2$ are isolated singularities of $f(z)$.

$$f(z) = \frac{A}{z-1} + \frac{B}{z+2} + \frac{C}{(z+2)^2}$$

$$z^2 = A(z+2)^2 + B(z-1)(z+2) + C(z-1).$$

Put $z=1 \Rightarrow A = 1/9$

Put $z=-2 \Rightarrow C = -4/3$

Put $z=0 \Rightarrow B = 8/9$

$$f(z) = \frac{1}{9(z-1)} + \frac{8}{9(z+2)} - \frac{4}{3(z+2)^2} \longrightarrow (1)$$

Case (i) :

To find : $[\text{Res } f(z)]_{z=1}$

Put $z-1 = u \Rightarrow z = u+1$

$$(1) \Rightarrow f(z) = \frac{1}{9u} + \frac{8}{9(u+1+2)} - \frac{4}{3(u+1+2)^2}$$

$$= \frac{1}{9u} + \frac{8}{9(u+3)} - \frac{4}{3(u+3)^2}$$

$$= \frac{1}{9u} + \frac{8}{9(3)\left(1+\frac{u}{3}\right)} - \frac{4}{3\cdot 9\left(1+\frac{u}{3}\right)^2}$$

$$= \frac{1}{9u} + \frac{8}{27}\left(1+\frac{u}{3}\right)^{-1} - \frac{4}{27}\left(1+\frac{u}{3}\right)^{-2}$$

$$= \frac{1}{9u} + \frac{8}{27}\left[1 - \frac{u}{3} + \left(\frac{u}{3}\right)^2 - \left(\frac{u}{3}\right)^3 + \cdots\right]$$

$$- \frac{4}{27}\left[1 - 2\left(\frac{u}{3}\right) + 3\left(\frac{u}{3}\right)^2 - 4\left(\frac{u}{3}\right)^3 + \cdots\right]$$

$$= \frac{1}{9u} + \frac{8}{27} - \frac{8}{27}\left(\frac{u}{3}\right) + \frac{8}{27}\left(\frac{u}{3}\right)^2 - \frac{8}{27}\left(\frac{u}{3}\right)^3 + \cdots$$

$$- \frac{4}{27} + \frac{8}{27}\left(\frac{u}{3}\right) - \frac{12}{27}\left(\frac{u}{3}\right)^2 + \frac{16}{27}\left(\frac{u}{3}\right)^3 + \cdots$$

$$= \frac{1}{9u} + \frac{4}{27} - \frac{4}{27}\left(\frac{u}{3}\right)^2 + \frac{8}{27}\left(\frac{u}{3}\right)^3 + \cdots$$

$$= \frac{1}{9(z-1)} + \frac{4}{27} - \frac{4}{27}\left(\frac{z-1}{3}\right)^2 + \frac{8}{27}\left(\frac{z-1}{3}\right)^3 + \cdots \rightarrow (2)$$

$$\left[\operatorname{Res} f(z)\right]_{z=1} = \text{co-eff. of } \frac{1}{z-1} = \frac{1}{9}$$

Case (ii) : To find $\left[\operatorname{Res} f(z)\right]_{z=-2}$

Put $z+2 = u \Rightarrow z = u-2$

$$(2) \Rightarrow f(z) = \frac{1}{9(u-2-1)} + \frac{8}{9u} - \frac{4}{3u^2}$$

$$= \frac{1}{9(u-3)} + \frac{8}{9u} - \frac{4}{3u^2}$$

$$= \frac{1}{-27\left(1-\frac{u}{3}\right)} + \frac{8}{9u} - \frac{4}{3u^2}$$

$$= -\frac{1}{27}\left(1-\frac{u}{3}\right)^{-1} + \frac{8}{9u} - \frac{4}{3u^2}$$

$$= -\frac{1}{27}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3 + \cdots\right] + \frac{8}{9(z+2)} - \frac{4}{3(z+2)^2}$$

$$= -\frac{1}{27}\left[1 + \left(\frac{z+2}{3}\right) + \left(\frac{z+2}{3}\right)^2 + \left(\frac{z+2}{3}\right)^3 + \cdots\right] + \frac{8}{9(z+2)} - \frac{4}{3(z+2)^2}$$

$$\left[\operatorname{Res} f(z)\right]_{z=-2} = \text{co-eff. of } \frac{1}{z+2} = \frac{8}{9}$$

Contour Integration :

Type – I :

$$\int_0^{2\pi} f(\theta)\, d\theta \quad , \quad |z| = 1$$

Take $z = e^{i\theta} \Rightarrow d\theta = \frac{dz}{iz}$

Replace $\cos\theta$ by $\frac{z^2+1}{2z}$

$\sin\theta$ by $\frac{z^2-1}{2iz}$