$f''(z) = \frac{-1}{(1+z)^2}$ $\qquad f''(0) = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\qquad f'''(0) = 2$

$f^{iv}(z) = \frac{-6}{(1+z)^4}$ $\qquad f^{iv}(0) = -6$

Taylor's series about $z = 0$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \dots$$

$$= 0 + \frac{1}{1!} z + \frac{(-1)}{2} z^2 + \frac{2}{6} z^3 + \frac{(-6)}{24} z^4 + \dots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots$$

Laurent's Series :

Let C_1, C_2 be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic on C_1 & C_2 and in the annular region R. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{1+n}} dz$

$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$.

Problems :

1) Find the Laurent's series expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ valid in the region

(i) $1 < |z| < 2$ $\qquad$ (ii) $|z| > 2$ and $0 < |z-1| < 1$.

Soln :

Given, $f(z) = \frac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

Put $z = 1 \Rightarrow A = -1$

Put $z = 2 \Rightarrow B = 1$

$$f(z) = \frac{1}{(z-1)(z-2)} = \frac{-1}{z-1} + \frac{1}{z-2}$$

(ii) Given $|z| > 2 \Rightarrow 2 < |z|$

$$\Rightarrow 1 > \frac{2}{|z|} \Rightarrow \frac{2}{|z|} < 1$$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z(1-1/z)} + \frac{1}{z(1-2/z)}$$

$$= -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} + \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f(z) = -\frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] + \frac{1}{z}\left[1+\frac{2}{z}+\left(\frac{2}{z}\right)^2+\cdots\right]$$

(i) $1 < |z| < 2$

$|z| < 2 \quad , \quad 1 < |z|$

$\frac{|z|}{2} < 1 \quad , \quad \frac{1}{|z|} < 1$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z\left(1-\frac{1}{z}\right)} + \frac{1}{-2\left(1-\frac{z}{2}\right)}$$

$$= \frac{-1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

$$= \frac{-1}{z}\left[1+\frac{1}{z}+\left(\frac{1}{z}\right)^2+\left(\frac{1}{z}\right)^3+\cdots\right] - \frac{1}{2}\left[1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\cdots\right]$$

(iii) Given $0 < |z-1| < 1$

Let $u = z - 1$

$0 < |u| < 1 \Rightarrow |u| < 1$

$$f(z) = \frac{-1}{z-1} + \frac{1}{z-2} = \frac{-1}{z-1} + \frac{1}{(z-1)-1}$$

$$= -\frac{1}{u} + \frac{1}{u-1} = -\frac{1}{u} - \frac{1}{1-u} = -\frac{1}{u} - (1-u)^{-1}$$

$$= \frac{-1}{u} - \left[1+u+u^2+u^3+\cdots\right]$$

Put $u = z - 1$

$$f(z) = -\frac{1}{z-1} - \left[1+(z-1)+(z-1)^2+(z-1)^3+\cdots\right]$$

2) Find the Laurent's series expansion of $f(z) = \dfrac{1}{(z+1)(z+3)}$ valid in $|z| > 3$, $1 < |z| < 3$.

<u>Soln</u>:

Given, $f(z) = \dfrac{1}{(z+1)(z+3)}$

$$\frac{1}{(z+1)(z+3)} = \frac{A}{z+1} + \frac{B}{z+3} = \frac{A(z+3) + B(z+1)}{(z+1)(z+3)}$$

$$1 = A(z+3) + B(z+1).$$

Put $z = -3 \Rightarrow B = -1/2$

Put $z = -1 \Rightarrow A = 1/2$

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{2}\frac{1}{z+1} - \frac{1}{2}\frac{1}{z+3}.$$

(i) $|z| > 3 \Rightarrow 1 > \dfrac{3}{|z|} \Rightarrow \dfrac{3}{|z|} < 1$

$$f(z) = \frac{1}{2}\frac{1}{z\left(1+\frac{1}{z}\right)} - \frac{1}{2}\frac{1}{z\left(1+\frac{3}{z}\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right]$$

$$- \frac{1}{2z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$

(ii) $1 < |z| < 3$

$1 < |z| \qquad |z| < 3$

$\dfrac{1}{|z|} < 1 \qquad \dfrac{|z|}{3} < 1$

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{6\left(\frac{z}{3}+1\right)}$$

$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \left(\frac{1}{z}\right)^2 - \left(\frac{1}{z}\right)^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \left(\frac{z}{3}\right)^3 + \cdots\right]$$

3) Obtain the Laurent's series expansion of $f(z) = \dfrac{z^2-1}{(z+2)(z+3)}$ in $2 < |z| < 3$ & $|z| > 3$.

Soln:

$$f(z) = \frac{z^2-1}{(z+2)(z+3)} = A + \frac{B}{z+2} + \frac{C}{z+3}$$

$$\frac{z^2-1}{(z+2)(z+3)} = \frac{A(z+2)(z+3) + B(z+3) + C(z+2)}{(z+2)(z+3)}$$

$$z^2 - 1 = A(z+2)(z+3) + B(z+3) + C(z+2).$$

Put $z = -2 \Rightarrow B = 3$

Put $z = -3 \Rightarrow C = -8$

Put $z = 0 \Rightarrow -1 = 6A + 3B + 2C$

$\Rightarrow -1 = 6A + 9 - 16$

$\Rightarrow A = 1$

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}.$$

(i) $2 < |z| < 3$

$|z| < 3 \qquad 2 < |z|$

$\frac{|z|}{3} < 1 \qquad \frac{2}{|z|} < 1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{3\left(\frac{z}{3}+1\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{3}\left[1 - \frac{z}{3} + \left(\frac{z}{3}\right)^2 - \cdots\right]$$

(ii) $|z| > 3 \Rightarrow 3 < |z| \Rightarrow \frac{3}{|z|} < 1$

$$f(z) = 1 + \frac{3}{z\left(1+\frac{2}{z}\right)} - \frac{8}{z\left(1+\frac{3}{z}\right)}$$

$$= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= 1 + \frac{3}{z}\left[1 - \frac{2}{z} + \left(\frac{2}{z}\right)^2 - \left(\frac{2}{z}\right)^3 + \cdots\right] - \frac{8}{z}\left[1 - \frac{3}{z} + \left(\frac{3}{z}\right)^2 - \left(\frac{3}{z}\right)^3 + \cdots\right]$$

4) Expand $f(z) = \frac{7z-2}{z(z-2)(z+1)}$ as Laurant's series in $1 < |z+1| < 3$.

Soln:

$$f(z) = \frac{7z-2}{z(z-2)(z+1)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z+1}$$

$$= \frac{A(z-2)(z+1) + Bz(z+1) + Cz(z-2)}{z(z-2)(z+1)}$$

$$7z-2 = A(z-2)(z+1) + Bz(z+1) + Cz(z-2)$$

Put $z = 0 \Rightarrow A = 1$

Put $z = 2 \Rightarrow B = 2$

Put $z = -1 \Rightarrow C = -3$

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

Given, $1 < |z+1| < 3$.

put $z+1 = u$

$1 < |u| < 3$

$1 < |u| \qquad |u| < 3$

$\frac{1}{|u|} < 1 \qquad \frac{|u|}{3} < 1$.

$$f(z) = \frac{1}{z} + \frac{2}{z-2} - \frac{3}{z+1}$$

$$= \frac{1}{u-1} + \frac{2}{u-1-2} - \frac{3}{u-1+1} = \frac{1}{u-1} + \frac{2}{u-3} - \frac{3}{u}$$

$$= -\frac{3}{u} + \frac{1}{u\left(1-\frac{1}{u}\right)} + \frac{2}{3\left(\frac{u}{3}-1\right)}$$

$$= -\frac{3}{u} + \frac{1}{u}\left(1-\frac{1}{u}\right)^{-1} - \frac{2}{3}\left(1-\frac{u}{3}\right)^{-1}$$

$$= -\frac{3}{u} + \frac{1}{u}\left[1 + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \left(\frac{1}{u}\right)^3 + \cdots\right] - \frac{2}{3}\left[1 + \left(\frac{u}{3}\right) + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3 + \cdots\right]$$

$$= -\frac{3}{u} + \frac{1}{u} + \left(\frac{1}{u}\right)^2 + \left(\frac{1}{u}\right)^3 + \left(\frac{1}{u}\right)^4 + \cdots - \frac{2}{3}\left[1 + \frac{u}{3} + \left(\frac{u}{3}\right)^2 + \left(\frac{u}{3}\right)^3 + \cdots\right]$$