$(z-1)(z-2) = 0$

$z = 1, 2$

$|z-2| = |1-2| = |-1| = 1 > 1/2$

$\therefore z = 1$ lies outside $C$.

$|z-2| = |2-2| = 0 < 1/2$

$\therefore z = 2$ lies inside $C$.

$$\int_C \frac{z\,dz}{(z-1)(z-2)^2} = \int_C \frac{\frac{z}{z-1}}{(z-2)^2}\,dz$$

$$= 2\pi i\, f'(2)$$

$$= 2\pi i \left[\frac{(z-1)(1) - z(1)}{(z-1)^2}\right]_{z=2}$$

$$= 2\pi i \left[\frac{1-2}{1^2}\right] = 2\pi i(-1) = -2\pi i.$$

**Taylor's Series :**

$$f(z) = f(a) + (z-a)\frac{f'(a)}{1!} + (z-a)^2\frac{f''(a)}{2!} + \cdots$$

This is known as Taylor's series of $f(z)$ at $z = a$.

**Maclaurin's Series :**

Put $a = 0$ in the Taylor series for $f(z)$ then

$$f(z) = f(0) + \frac{f'(0)}{1!}z + \frac{f''(0)}{2!}z^2 + \frac{f'''(0)}{3!}z^3 + \cdots$$

This series is called Maclaurin's series of $f(z)$.

**Problems :**

1) Expand $f(z) = \sin z$ in a Taylor series about $z = 0$.

**Soln :**

| Function | At $z = 0$ |
|---|---|
| $f(z) = \sin z$ | $f(0) = 0$ |
| $f'(z) = \cos z$ | $f'(0) = 1$ |
| $f''(z) = -\sin z$ | $f''(0) = 0$ |
| $f'''(z) = -\cos z$ | $f'''(0) = -1$ |
| $f^{iv}(z) = \sin z$ | $f^{iv}(0) = 0$ |

Taylor series about $z = 0$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

$$= 0 + \frac{1 \cdot z}{1!} + \frac{0 \cdot z^2}{2!} + \frac{(-1) z^3}{3!} + 0 + \cdots$$

$$= \frac{z}{1!} - \frac{z^3}{3!} + \cdots$$

2) Expand $\frac{z-1}{z+1}$ in Taylor series about the point $z = 1$.

Soln:

| Function | At $z = 1$ |
|---|---|
| $f(z) = \frac{z-1}{z+1}$ | $f(1) = 0$ |
| $f'(z) = \frac{(z+1)-(z-1)}{(z+1)^2} = \frac{2}{(z+1)^2}$ | $f'(1) = 1/2$ |
| $f''(z) = \frac{-4}{(z+1)^3}$ | $f''(1) = -1/2$ |
| $f'''(z) = \frac{12}{(z+1)^4}$ | $f'''(1) = 3/4$ |

Taylor series about $z = 1$

$$f(z) = f(1) + (z-1) \frac{f'(1)}{1!} + \frac{(z-1)^2}{2!} f''(1) + \cdots$$

$$= 0 + \frac{(z-1)}{1!}\left(\frac{1}{2}\right) + \frac{(z-1)^2}{2}\left(-\frac{1}{2}\right) + \frac{(z-1)^3}{6}\left(\frac{3}{4}\right) + \cdots$$

$$= \frac{z-1}{2} - \frac{(z-1)^2}{4} + \frac{(z-1)^3}{8} + \cdots$$

3) Expand $\log(1+z)$ as a Taylor's series about $z = 0$.

Soln:

| Function | At $z = 0$. |
|---|---|
| $f(z) = \log(1+z)$ | $f(0) = \log 1 = 0$ |
| $f'(z) = \frac{1}{1+z}$ | $f'(0) = 1.$ |

$f''(z) = \frac{-1}{(1+z)^2}$ $\quad f''(0) = -1$

$f'''(z) = \frac{2}{(1+z)^3}$ $\quad f'''(0) = 2$

$f^{IV}(z) = \frac{-6}{(1+z)^4}$ $\quad f^{IV}(0) = -6$

Taylor's series about $z = 0$

$$f(z) = f(0) + \frac{f'(0)}{1!} z + \frac{f''(0)}{2!} z^2 + \frac{f'''(0)}{3!} z^3 + \cdots$$

$$= 0 + \frac{1}{1!} z + \frac{(-1)}{2} z^2 + \frac{2}{6} z^3 + \frac{(-6)}{24} z^4 + \cdots$$

$$= z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$$

**Laurent's Series :**

Let $C_1, C_2$ be two concentric circles $|z-a| = R_1$ and $|z-a| = R_2$ where $R_2 < R_1$.

Let $f(z)$ be analytic on $C_1$ & $C_2$ and in the annular region $R$. Then

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-a)^n}$$

where $a_n = \frac{1}{2\pi i} \int_{C_1} \frac{f(z)}{(z-a)^{1+n}} dz$

$b_n = \frac{1}{2\pi i} \int_{C_2} \frac{f(z)}{(z-a)^{1-n}} dz$.

**Problems :**

1) Find the Laurent's series expansion of $f(z) = \frac{1}{(z-1)(z-2)}$ valid in the region

(i) $1 < |z| < 2$ (ii) $|z| > 2$ and $0 < |z-1| < 1$.

Soln :

Given, $f(z) = \frac{1}{(z-1)(z-2)}$

$$\frac{1}{(z-1)(z-2)} = \frac{A}{z-1} + \frac{B}{z-2} = \frac{A(z-2) + B(z-1)}{(z-1)(z-2)}$$

$$1 = A(z-2) + B(z-1)$$