Harmonic Function

A function $u(x,y)$ is called harmonic when $u_{xx} + u_{yy} = 0$.

(1) Verify whether the function $u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$ is harmonic.

Verification

$$u = x^3 - 3xy^2 + 3x^2 - 3y^2 + 1$$

$u_x = 3x^2 - 3y^2 + 6x$ $\qquad$ $u_y = -6xy - 6y$

$u_{xx} = 6x + 6$ $\qquad$ $u_{yy} = -6x - 6$

$u_{xx} + u_{yy} = 6x + 6 - 6x - 6 = 0$

$\Rightarrow$ u is harmonic.

(2) Show that $u = 2x - x^3 + 3xy^2$ is harmonic.

Sol

$$u = 2x - x^3 + 3xy^2$$

Properties of Analytic Function

① Prove that the real and imaginary parts of an analytic function are harmonic.

proof

Take $f(z) = u + iv$ be analytic.

Then $u_x = v_y$; $v_x = -u_y$
$\quad\quad \hookrightarrow ① \quad\quad\quad \hookrightarrow ②$

Diff ① w.r.t. x, $u_{xx} = v_{xy}$

Diff ② w.r.t. y, $v_{yx} = -u_{yy}$

$$\Rightarrow u_{yy} = -v_{yx}$$

$$\therefore u_{xx} + u_{yy} = v_{xy} - v_{yx} = 0$$

$\Rightarrow u$ is harmonic.

Similarly, v is harmonic.

Note

From the above property,

Proof

$$u = x^2 - y^2$$

$$u_x = 2x \qquad u_y = -2y$$

$$u_{xx} = 2 \qquad u_{yy} = -2$$

$$u_{xx} + u_{yy} = 0 \Rightarrow u \text{ is harmonic.}$$

$$v = \frac{-y}{x^2+y^2} = -y\,(x^2+y^2)^{-1}$$

$$v_x = y\,(x^2+y^2)^{-2} \cdot 2x = \frac{2xy}{(x^2+y^2)^2}$$

$$v_{xx} = \frac{(x^2+y^2)^2 \cdot 2y - 2xy \cdot 2(x^2+y^2) \cdot 2x}{(x^2+y^2)^4}$$

$$= \frac{2y(x^2+y^2)\left[\cancel{2x^2y} + \cancel{2y^3} - x^2 + y^2 - 4x^2\right]}{(x^2+y^2)^4}$$

$$v_{xx} = \frac{2y\,(y^2-3x^2)}{(x^2+y^2)^3}$$

$$v_y = \frac{-(x^2+y^2) + y\,(2y)}{\ldots} = \frac{y^2-x^2}{\ldots}$$

$$= \left[\frac{2y(x^2+y^2)\left[x^2+y^2-2y^2+2x^2\right]}{(x^2+y^2)^4}\right]$$

$$V_{yy} = \frac{2y\,(3x^2-y^2)}{(x^2+y^2)^3}$$

$V_{xx} + V_{yy} = 0 \Rightarrow V$ is harmonic.

$U_x \neq V_y \Rightarrow$ C-R equations are not satisfied.

$\Rightarrow f(z) = u+iv$ is not analytic.

(2) Prove that an analytic function with constant real part (Imaginary part) is constant.

Proof

Let $f(z) = u+iv$ be analytic function.

Then $u_x = V_y$; $V_x = -u_y$.

Take $u =$ Constant

3) Prove that an analytic function with Constant modulus is constant.

Proof

Let $f(z) = u+iv$ be analytic.

Then $u_x = v_y$; $v_x = -u_y$.

$|f(z)| = \sqrt{u^2+v^2}$.

Given Modulus = Constant

$\Rightarrow \sqrt{u^2+v^2} = c$

$\Rightarrow u^2+v^2 = c^2$.

Diff. w.r.t. x

$2uu_x + 2vv_x = 0$

$uu_x + vv_x = 0 \rightarrow (1)$

Diff. w.r.t. y

$2uu_y + 2vv_y = 0$.

$-uv_x + vu_x = 0$. $\rightarrow (2)$

$(1) \times v \Rightarrow uv\,u_x + v^2 v_x = 0$.

(4) If $f(z) = u+iv$ is analytic, prove that the curves $u =$ Constant & $v =$ Constant are orthogonal.

Proof

Given $f(z) = u+iv$ is analytic

$\Rightarrow \quad u_x = v_y \quad ; \quad v_x = -u_y$

Given $u(x,y) =$ Constant — (1)

Slope of (1) is $\boxed{m_1 = \frac{-u_x}{u_y}}$

$v(x,y) =$ Constant — (2)

Slope of (2) is $m_2 = \frac{-v_x}{v_y}$

$$m_1 m_2 = -\frac{u_x}{u_y} \times \frac{-v_x}{v_y} = \frac{u_x}{u_y} \times \frac{-u_y}{u_x} = -1$$

$\Rightarrow$ The curves are orthogonal.

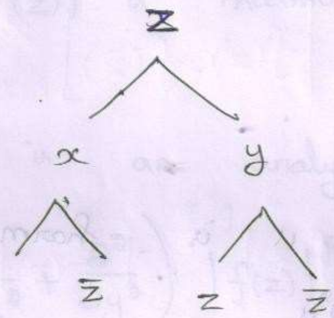
Result

$$\frac{\partial^2}{} + \frac{\partial^2}{}$$

$$x = \frac{z+\bar{z}}{2} \qquad y = \frac{z-\bar{z}}{2i}$$

$$\frac{\partial x}{\partial z} = \frac{1}{2} \qquad \frac{\partial y}{\partial z} = \frac{1}{2i}$$

$$\frac{\partial x}{\partial \bar{z}} = \frac{1}{2} \qquad \frac{\partial y}{\partial \bar{z}} = \frac{-1}{2i}$$



$$\frac{\partial}{\partial z} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}$$

$$\Rightarrow \boxed{\frac{\partial}{\partial z} = \frac{1}{2}\frac{\partial}{\partial x} + \frac{1}{2i}\frac{\partial}{\partial y}}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}} + \frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}$$

$$\boxed{\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\frac{\partial}{\partial x} - \frac{1}{2i}\frac{\partial}{\partial y}}$$

$$\Rightarrow \boxed{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}}$$

(17) Show that a harmonic function u Satisfies the formal differential equation $\frac{\partial^2 u}{\partial z \partial \bar{z}} = 0$.

proof

Given u is harmonic.

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

WKT $4\frac{\partial^2}{\partial z \partial \bar{z}} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$

$$\Rightarrow \frac{\partial^2 u}{\partial z \partial \bar{z}} = \frac{1}{4}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0.$$

If $f(z)$ is a regular function, P.T. $\log|f'(z)|$ is harmonic.

proof

$$= 4 \left\{ \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ \log f(z) + \log f'(\bar{z}) \right\} \right.$$

$$= 4 \left\{ \frac{\partial}{\partial z} \left[\frac{f''(\bar{z})}{f'(\bar{z})} \right] \right\} = 0.$$

$\Rightarrow \log |f'(z)|$ is harmonic.

(2) If $f(z)$ is an analytic function, Prove that $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p = p^2 |f'(z)|^2 |f(z)|^{p-2}$.

Proof

WKT $\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = 4 \frac{\partial^2}{\partial z \partial \bar{z}}$

$$LHS = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ |f(z)|^2 \right\}^{p/2}$$

$$= 4 \frac{\partial^2}{\partial z \partial \bar{z}} \left\{ f(z) \cdot f(\bar{z}) \right\}^{p/2}$$

$$= 4 \frac{\partial^2}{\partial z} \left\{ f(z)^{p/2} \cdot (p/2)\, f(\bar{z})^{p/2 - 1} \cdot f'(\bar{z}) \right\}$$

$$= 4 \left\{ p/2 \cdot f(z)^{p/2 - 1} \cdot f'(z) \cdot p/2 \cdot f(\bar{z})^{p/2 - 1} \cdot f'(\bar{z}) \right\}$$

$$= \frac{4 \times p^2}{4} \left[f(z) \cdot f(\bar{z}) \right]^{\frac{p-2}{2}} \cdot \left[f'(z) \cdot \overline{f'(z)} \right]$$

$$= p^2 \, |f(z)|^{p-2} \cdot |f'(z)|^2$$

$$\therefore \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^p = p^2 \, |f(z)|^{p-2} \cdot |f'(z)|^2.$$

When $p = 2$,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) |f(z)|^2 = 4 \, |f'(z)|^2.$$

Construction of Analytic Functions

Milne - Thompson Method