UNIT –IV

COMPLEX INTEGRATION

4.1 LINE INTEGRAL AND CONTOUR INTEGRAL

If $f(z)$ is a continuous function of the complex variable $z = x + iy$ and C is any continuous curve connecting two points A and B on the z – plane then the complex line integral of $f(z)$ along C from A to B is denoted by $\int_c f(z)dz$

When C is simple closed curve, then the complex integral is also called as a contour integral and is denoted as $\oint_C f(z)dz$. The curve C is always take in the anticlockwise direction.

Note: If the direction of C is reversed (clockwise), the integral changes its sign

$$(ie) \oint_c f(z)dz = -\oint f(z)dz$$

Standard theorems:

1. **Cauchy's Integral theorem (or) Cauchy's Theorem (or) Cauchy's Fundamental Theorem**

   Statement: If $f(z)$ is analytic and its derivative $f'(z)$ is continuous at all points inside and on a simple closed curve C then $\oint_c f(z)\, dz = 0$

2. **Extension of Cauchy's integral theorem (or) Cauchy's theorem for multiply connected Region**

   Statement: If $f(z)$ is analytic at all points inside and on a multiply connected region whose outer boundary is C and inner boundaries are $C_1, C_2, \dots, C_n$ then

$$\int_c f(z)dz = \int_c f(z)dz + \int_{C_2} f(z)dz + \cdots + \int_{C_n} f(z)dz$$

3. **Cauchy's integral formula**

Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected region R and if 'a' is any point interior to C, then

$$f(a) = \frac{1}{2\pi i} \int_c \frac{f(z)}{z-a} dz$$

(OR)

$$\int_c \frac{f(x)}{z-a} dz = 2\pi i\, f(a),$$

the integration around C being taken in the positive direction.

4. **Cauchy's Integral formula for derivatives**

Statement: If $f(z)$ is analytic inside and on a simple closed curve C of a simply connected Region R and if 'a' is any point interior to C, then

$$\int_c \frac{f(z)}{(z-a)^2} dz = 2\pi i f'(a)$$

$$\int_c \frac{f(z)}{(z-a)^3} dz = 2\pi i f''(a)$$

In general, $\int_c \frac{f(z)}{(z-a)^n} dz = 2\pi i f^{(n-1)}(a)$

Problems based on Cauchy's Integral Theorem

Example: 4.1 Evaluate $\int_{o}^{3+i} z^2 dz$ along the line joining the points (0, 0) and (3, 1)

Solution:

Given $\int_0^{3+i} z^2 dz$

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 3 + i$ corresponds to (3, 1)

The equation of the line joining (0, 0) and (3, 1) is

$$y = \frac{x}{3} \Rightarrow x = 3y$$

Now $z^2 dz = (x + iy)^2(dx + idy)$

$$= [x^2 - y^2 + i2xy][dx + idy]$$

$$= [(x^2 - y^2) + i2xy][dx + idy]$$

$$= [(x^2 - y^2)dx - 2xydy] + i[2xydx + (x^2 - y^2)dy]$$

Since $x = 3y \Rightarrow dx = 3dy$

$$\therefore z^2 dz = [8y^2(3dy) - 6y^2dy] + i[18y^2dy + 8y^2dy]$$

$$= 18y^2dy + i26y^2dy$$

$$\therefore \int_0^{3+i} z^2 dz = \int_0^{'} [18y^2 + i26y^2]dy$$

$$= \left[18\ \frac{y^2}{3} + i\ 26\ \frac{y^3}{3}\right]_0^{'}$$

$$= 6 + i\frac{26}{3}$$

Example: 4.2 Evaluate $\int_0^{2+i} (x^2 - iy)dz$

Solution:

Let $z = x + iy$

Here $z = 0$ corresponds to (0, 0) and $z = 2 + i$ corresponds to (2, 1)

Now $(x^2 - iy)dz = (x^2 - iy)(dx + idy)$

$$= x^2dx + y\ dy) + i\ (x^2dy - y\ dx)$$

Along the path $y = x^2 \Rightarrow dy = 2xdx$

$$\therefore (x^2 - iy)dz = (x^2dx + 2x^3dx) + i(2x^3dx - x^2dx)$$

$$\int_0^{2+i} (x^2 - iy)dz = \int_0^2 (x^2 + 2x^3)dx + i(2x^3 - x^2)dx$$

$$= \left[\frac{x^3}{3} + \frac{2x^4}{4}\right]_0^2 + i\left[\frac{2x^4}{4} = \frac{x^3}{3}\right]_0^2$$

$$= \left(\frac{8}{3} + \frac{16}{2}\right) + i\ \left(\frac{16}{2} - \frac{8}{3}\right)$$

$$= \frac{32}{3} + i\ \frac{16}{3}$$

Example: 4.3 Evaluate $\int_c e^{\frac{1}{z}} dz, where\ C\ is\ |z| = 2$

Solution:

Let $f(z) = e^{\frac{1}{z}}$ clearly $f(z)$ is analytic inside and on C.

Hence, by Cauchy's integral theorem we get $\int_c e^{\frac{1}{z}} dz = 0$

Example: 4.4 Evaluate $\int_c z^2 e^{\frac{1}{z}} dz, where\ C\ is\ |z| = 1$

Solution:

$$\text{Given} \int_c z^2 e^{1/z} dz$$

$$= \int_c \frac{z^2}{e^{-1/z}} dz$$

$Dr = 0 \implies z = 0$, We get $e^{-\frac{1}{0}} = e^{-\infty} = 0$

$z = 0$ lies inside $|z| = 1$.

Cauchy's Integral formula is

$$\int_c z^2 e^{1/z} dz = 2\pi i f(0) = 0$$

Example: 4.5 Evaluate $\int_c \frac{1}{2z-3} dz\ where\ C\ is\ |z| = 1$

Solution:

$$\text{Given} \int_c \frac{1}{2z-3} dz$$

$Dr = 0 \implies 2z - 3 = 0, \implies z = \frac{3}{2}$

Given $C\ is\ |z| = 1$

$$\Rightarrow |z| = \left|\frac{3}{2}\right| = \frac{3}{2} > 1$$

$\therefore z = \frac{3}{2}$ lies outside C

$\therefore$ By Cauchy's Integral theorem, $\int_c \frac{1}{2z-3} dz = 0$

Example: 4.6 Evaluate $\int_c \frac{dz}{z+4}$ $\boldsymbol{where\ C\ is\ |z| = 2}$

Solution:

Given $\int_c \frac{dz}{z+4}$

$Dr = 0 \implies z + 4 = 0 \implies z = -4$

Given $C\ is\ |z| = 2$

$\Rightarrow |z| = |-4| = 4 > 2$

$\therefore z = -4$ lies outside C.

$\therefore$ By Cauchy's Integral Theorem, $\int_c \frac{dz}{z+4} = 0$

Example: 4.7 Evaluate $\int_c \frac{e^{2z}}{z^2+1} dz, \boldsymbol{where\ C\ is\ |z| = \frac{1}{2}}$

Solution:

Given $\int_c \frac{e^{2z}}{z^2+1} dz$

$Dr = 0 \implies z^2 + 1 = 0 \implies z = \pm i$

Given $C\ is\ |z| = \frac{1}{2}$

$\Rightarrow |z| = |\pm i| = 1 > \frac{1}{2}$

$\therefore$Clearly both the points $z = \pm i$ lies outside C.

$\therefore$ By Cauchy's Integral Theorem, $\int_c \frac{e^{2z}}{z^2+1} dz = 0$