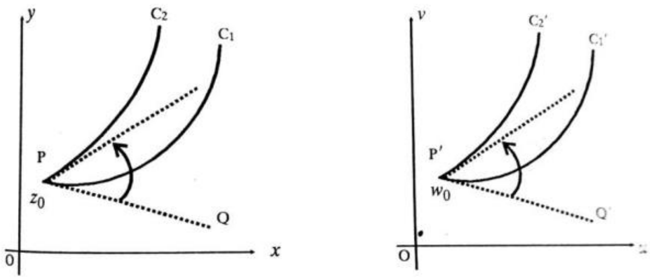
3.5 CONFORMAL MAPPING

Definition: Conformal Mapping

A transformation that preserves angels between every pair of curves through a point, both in magnitude and sense, is said to be conformal at that point.



Definition: Isogonal

A transformation under which angles between every pair of curves through a point are preserved in magnitude, but altered in sense is said to be an isogonal at that point.

Note: 3.4 (i) A mapping $w = f(z)$ is said to be conformal at $z = z_0$, if $f'(z_0) \neq 0$.

Note: 3.4 (ii) The point, at which the mapping $w = f(z)$ is not conformal,

$(i.e.) f'(z) = 0$ is called a **critical point** of the mapping.

If the transformation $w = f(z)$ is conformal at a point, the inverse transformation $z = f^{-1}(w)$ is also conformal at the corresponding point.

The critical points of $z = f^{-1}(w)$ are given by $\frac{dz}{dw} = 0$. hence the critical point of the transformation $w = f(z)$ are given by $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$,

Note: 3.4 (iii) Fixed points of mapping.

Fixed or invariant point of a mapping $w = f(z)$ are points that are mapped onto themselves, are "Kept fixed" under the mapping. Thus they are obtained from $w = f(z) = z$.

The identity mapping $w = z$ has every point as a fixed point. The mapping $w = \bar{z}$ has infinitely many fixed points.

$w = \frac{1}{z}$ has two fixed points, a rotation has one and a translation has none in the complex plane.

Some standard transformations

Translation:

The transformation $w = C + z$, where C is a complex constant, represents a translation.

Let $z = x + iy$

$w = u + iv$ and $C = a + ib$

Given $w = z + C$,

$(i.e.)\ u + iv = x + iy + a + ib$

$\Rightarrow u + iv = (x + a) + i(y + b)$

Equating the real and imaginary parts, we get $u = x + a, v = y + b$

Hence the image of any point $p(x, y)$in the z−plane is mapped onto the point $p'(x + a, y + b)$ in the w-plane. Similarly every point in the z−plane is mapped onto the w plane.

If we assume that the w−plane is super imposed on the z-plane, we observe that the point (x, y) and hence any figure is shifted by a distance $|C| = \sqrt{a^2 + b^2}$ in the direction of C i.e., translated by the vector representing C.

Hence this transformation transforms a circle into an equal circle. Also the corresponding regions in the z and w planes will have the same shape, size and orientation.

Problems based on $w = z + k$

Example: 3.36 What is the region of the w plane into which the rectangular region in the Z plane bounded by the lines $x = 0, y = 0,\ x = 1\ and\ y = 2$ is mapped under the transformation $w = z + (2 - i)$

Solution:

Given $w = z + (2 - i)$

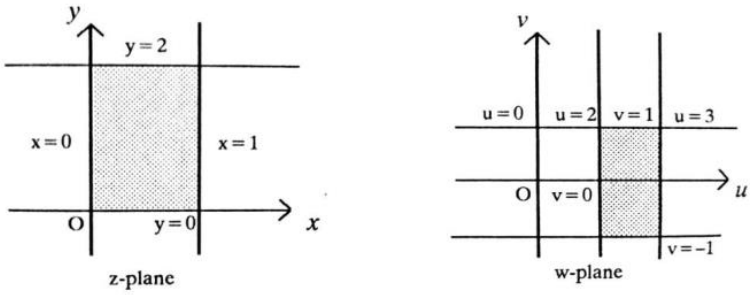
$(i.e.)\ u + iv = x + iy + (2 - i) = (x + 2) + i(y - 1)$

Equating the real and imaginary parts

$$u = x + 2,\ v = y - 1$$

Given boundary lines are	transformed boundary lines are
$x = 0$	$u = 0 + 2 = 2$
$y = 0$	$v = 0 - 1 = -1$
$x = 1$	$u = 1 + 2 = 3$
$y = 2$	$v = 2 - 1 = 1$

Hence, the lines $x = 0, y = 0, x = 1$, and $y = 2$ are mapped into the lines $u = 2, v = -1$, $u = 3, and\ v = 1$ respectively which form a rectangle in the w plane.



Example: 3.37 Find the image of the circle $|z| = 1$ by the transformation $w = z + 2 + 4i$

Solution:

Given $w = z + 2 + 4i$

$$(i.e.)\ u + iv = x + iy + 2 + 4i$$

$$= (x + 2) + i(y + 4)$$

Equating the real and imaginary parts, we get

$$u = x + 2, v = y + 4,$$

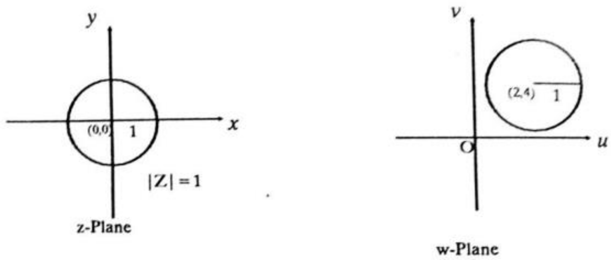
$$x = u - 2, y = v - 4,$$

Given $|z| = 1$

$$(i.e.)\ x^2 + y^2 = 1$$

$$(u - 2)^2 + (v - 4)^2 = 1$$

Hence, the circle $x^2 + y^2 = 1$is mapped into $(u - 2)^2 + (v - 4)^2 = 1$ in w plane which is also a circle with centre (2, 4)and radius 1.



2. Magnification and Rotation

The transformation $w = cz$, where c is a complex constant, represents both magnification and rotation.

This means that the magnitude of the vector representing z is magnified by $a = |c|$ and its direction is rotated through angle $\alpha = amp\ (c)$. Hence the transformation consists of a magnification and a rotation.

Problems based on w = cz

Example: 3.38 Determine the region 'D' of the w-plane into which the triangular region D enclosed by the lines $x = 0, y = 0, x + y = 1$ is transformed under the transformation $w = 2z$.

Solution:

Let $w = u + iv$

$z = x + iy$

Given $w = 2z$

$u + iv = 2(x + iy)$

$u + iv = 2x + i2y$

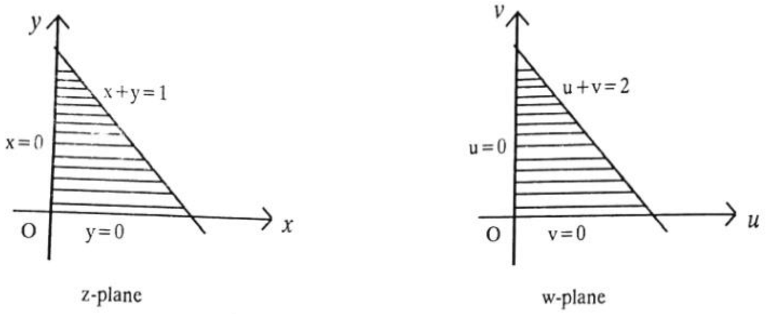
$u = 2x \Rightarrow x = \frac{u}{2}, v = 2y \Rightarrow y = \frac{v}{2}$

Given region (D) whose boundary lines are		Transformed region D' whose boundary lines are
$x = 0$	$\Rightarrow$	$u = 0$
$y = 0$	$\Rightarrow$	$v = 0$
$x + y = 1$	$\Rightarrow$	$\frac{u}{2} + \frac{v}{2} = 1 [\because x = \frac{u}{2}, y = \frac{v}{2}]$ $(i.e.)\ u + v = 2$

In the z plane the line $x = 0$ is transformed into $u = 0$in the w plane.

In the z plane the line $y = 0$ is transformed into $v = 0$in the w plane.

In the z plane the line $x + y =$ is transformed into$u + v = 2$ in the w plane.



Example: 3.39 Find the image of the circle $|z| = \lambda$ under the transformation $w = 5z$.

Solution:

Given $w = 5z$

$|w| = 5|z|$

i.e., $|w| = 5\lambda \quad [\because |z| = \lambda]$

Hence, the image of $|z| = \lambda$ in the z plane is transformed into $|w| = 5\lambda$ in the w plane under the transformation $w = 5z$.

Example: 3.40 Find the image of the circle $|z| = 3$ under the transformation $w = 2z$

[A.U N/D 2012] [A.U N/D 2016 R-13]

Solution:

Given $w = 2z,\ |z| = 3$

$|w| = (2)|z|$

$= (2)(3), \quad \text{Since } |z| = 3$

$= 6$

Hence, the image of $|z| = 3$ in the z plane is transformed into $|w| = 6$ w plane under the transformation $w = 2z$.

Example: 3.41 Find the image of the region $y > 1$ under the transformation

$w = (1 - i)z$. **[Anna, May – 1999]**

Solution:

Given $w = (1 - i)z$.

$u + v = (1 - i)(x + iy)$

$= x + iy - ix + y$

$= (x + y) + i(y - x)$

i.e., $u = x + y, \quad v = y - x$

$u + v = 2y \qquad u - v = 2x$

$y = \frac{u+v}{2} \qquad x = \frac{u-v}{2}$

Hence, image region $y > 1$ is $\frac{u+v}{2} > 1$ i.e., $u + v > 2$ in the w plane.

Problems based on $w = \frac{1}{z}$

Example: 3.42 Find the image of $|z - 2i| = 2$ under the transformation $w = \frac{1}{z}$

[Anna – May 1999, May 2001] [A.U N/D 2016 R-18]

Solution:

Given $|z - 2i| = 2$ .....(1) is a circle.

Centre = (0,2)

radius = 2

Given $w = \frac{1}{z} => z = \frac{1}{w}$

$$(1) \quad \Rightarrow \left|\frac{1}{w} - 2i\right| = 2$$

$$\Rightarrow |1 - 2wi| = 2|w|$$

$$\Rightarrow |1 - 2(u + iv)i| = 2|u + iv|$$

$$\Rightarrow |1 - 2ui + 2v| = 2|u + iv|$$

$$\Rightarrow |1 + 2v - 2ui| = 2|u + iv|$$

$$\Rightarrow \sqrt{(1 + 2v)^2 + (-2u)^2} = 2\sqrt{u^2 + v^2}$$

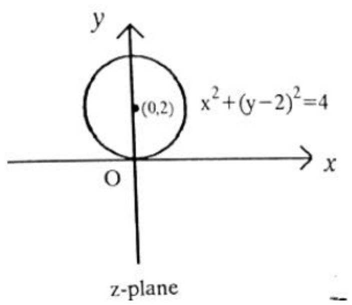
$$\Rightarrow (1 + 2v)^2 + 4u^2 = 4(u^2 + v^2)$$

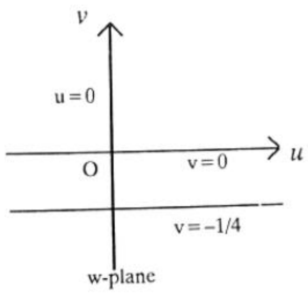
$$\Rightarrow 1 + 4v^2 + 4v + 4u^2 = 4(u^2 + v^2)$$

$$\Rightarrow 1 + 4v = 0$$

$$\Rightarrow v = -\frac{1}{4}$$

Which is a straight line in w plane.





Example: 3.43 Find the image of the circle $|z - 1| = 1$ in the complex plane under the mapping $w = \frac{1}{z}$

[A.U N/D 2009] [A.U M/J 2016 R-8]

Solution:

Given $|z - 1| = 1$ .....(1) is a circle.

Centre =(1,0)

radius = 1

Given $w = \frac{1}{z} \Rightarrow z = \frac{1}{w}$

$$(1) \quad \Rightarrow \left|\frac{1}{w} - 1\right| = 1$$

$$\Rightarrow |1 - w| = |w|$$

$$\Rightarrow |1 - (u + iv)| = |u + iv|$$

$$\Rightarrow |1 - u + iv| = |u + iv|$$

$$\Rightarrow \sqrt{(1 - u)^2 + (-v)^2} = \sqrt{u^2 + v^2}$$

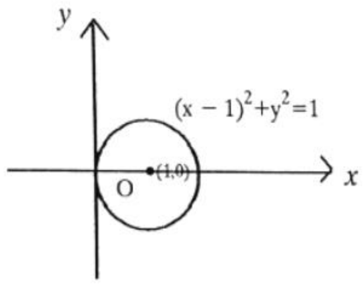
$$\Rightarrow (1 - u)^2 + v^2 = u^2 + v^2$$

$$\Rightarrow 1 + u^2 - 2v + v^2 = u^2 + v^2$$

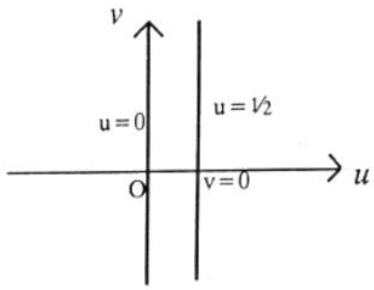
$$\Rightarrow 2u = 1$$

$$\Rightarrow u = \frac{1}{2}$$

which is a straight line in the w- plane



z-plane



w-plane

Example: 3.44 Find the image of the infinite strips

(i) $\frac{1}{4} < y < \frac{1}{2}$ (ii) $0 < y < \frac{1}{2}$ under the transformation $w = \frac{1}{z}$

Solution :

Given $w = \frac{1}{z}$ (given)

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \frac{u-iv}{u^2+v^2} = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

$$x = \frac{u}{u^2+v^2} \dots.(1), y = \frac{-v}{u^2+v^2} \dots.(2)$$

(i) Given strip is $\frac{1}{4} < y < \frac{1}{2}$

when $y = \frac{1}{4}$

$$\frac{1}{4} = \frac{-v}{u^2+v^2} \qquad \text{by (2)}$$

$$\Rightarrow u^2 + v^2 = -4v$$

$$\Rightarrow u^2 + v^2 + 4v = 0$$

$$\Rightarrow u^2 + (v+2)^2 = 4$$

which is a circle whose centre is at $(0, -2)$ in the w plane and radius is 2k.

when $y = \frac{1}{2}$

$$\frac{1}{2} = \frac{-v}{u^2+v^2} \qquad \text{by (2)}$$

$$\Rightarrow u^2 + v^2 = -2v$$

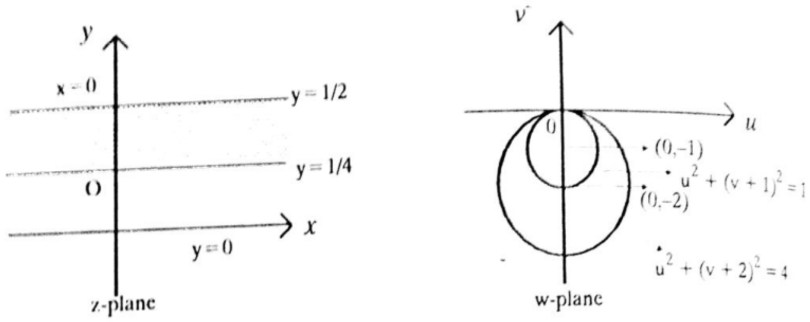
$$\Rightarrow u^2 + v^2 + 2v = 0$$

$$\Rightarrow u^2 + (v+1)^2 = 0$$

$$\Rightarrow u^2 + (v+1)^2 = 1 \qquad \dots\dots(3)$$

which is a circle whose centre is at $(0, -1)$ in the w plane and unit radius

Hence the infinite strip $\frac{1}{4} < y < \frac{1}{2}$ is transformed into the region in between circles $u^2 + (v + 1)^2 = 1$ and $u^2 + (v + 2)^2 = 4$ in the w plane.



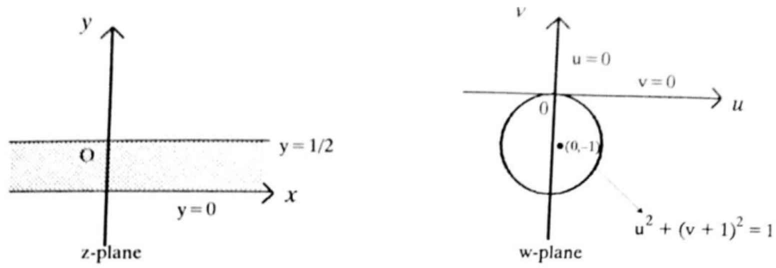
ii) Given strip is $0 < y < \frac{1}{2}$

when $y = 0$

$\Rightarrow v = 0$ by (2)

when $y = \frac{1}{2}$ we get $u^2 + (v + 1)^2 = 1$ by (3)

Hence, the infinite strip $0 < y < \frac{1}{2}$ is mapped into the region outside the circle $u^2 + (v + 1)^2 = 1$ in the lower half of the w plane.



Example: 3.45 Find the image of $x = 2$ under the transformation $w = \frac{1}{z}$. [Anna – May 1998]

Solution:

Given $w = \frac{1}{z}$

i.e., $z = \frac{1}{w}$

$$z = \frac{1}{u+iv} = \frac{u-iv}{(u+iv)+(u-iv)} = \frac{u-iv}{u^2+v^2}$$

$$x + iy = \left[\frac{u}{u^2+v^2}\right] + i\left[\frac{-v}{u^2+v^2}\right]$$

i.e., $x = \frac{u}{u^2+v^2} \ldots (1), y = \frac{-v}{u^2+v^2}$ ... (2)

Given $x = 2$ in the z plane.

$\therefore 2 = \frac{u}{u^2+v^2}$ by (1)

$$2(u^2 + v^2) = u$$

$$u^2 + v^2 - \frac{1}{2}u = 0$$

which is a circle whose centre is $\left(\frac{1}{4}, 0\right)$ and radius $\frac{1}{4}$

$\therefore x = 2$ in the z plane is transformed into a circle in the w plane.

Problems based on critical points of the transformation

Example: 3.54 Find the critical points of the transformation $w^2 = (z - \alpha)(z - \beta)$.

[A.U Oct., 1997] [A.U N/D 2014] [A.U M/J 2016 R-13]

Solution:

Given $w^2 = (z - \alpha)(z - \beta)$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow 2w\frac{dw}{dz} = (z-\alpha) + (z-\beta)$$

$$= 2z - (\alpha+\beta)$$

$$\Rightarrow \frac{dw}{dz} = \frac{2z-(\alpha+\beta)}{2w} \qquad \ldots(2)$$

Case $(i)\ \frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z-(\alpha+\beta)}{2w} = 0$$

$$\Rightarrow 2z - (\alpha+\beta) = 0$$

$$\Rightarrow 2z = \alpha+\beta$$

$$\Rightarrow z = \frac{\alpha+\beta}{2}$$

Case $(ii)\ \frac{dz}{dw} = 0$

$$\Rightarrow \frac{2w}{2z-(\alpha+\beta)} = 0$$

$$\Rightarrow \frac{w}{z-\frac{\alpha+\beta}{2}} = 0$$

$$\Rightarrow w = 0 \Rightarrow (z-\alpha)(z-\beta) = 0$$

$$\Rightarrow z = \alpha, \beta$$

$\therefore$ The critical points are $\frac{\alpha+\beta}{2}, \alpha$ and β.

Example: 3.55 Find the critical points of the transformation $w = z^2 + \frac{1}{z^2}$. [A.U A/M 2017 R-13]

Solution:

Given $w = z^2 + \frac{1}{z^2} \qquad \ldots(1)$

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 2z - \frac{2}{z^3} = \frac{2z^4-2}{z^3}$$

Case $(i)\ \frac{dw}{dz} = 0$

$$\Rightarrow \frac{2z^4-2}{z^3} = 0 \Rightarrow 2z^4 - 2 = 0$$

$$\Rightarrow z^4 - 1 = 0$$

$$\Rightarrow z = \pm 1, \pm i$$

Case $(ii)\ \frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{2z^4-2} = 0 \Rightarrow z^3 = 0 \Rightarrow z = 0$$

$\therefore$ The critical points are $\pm 1, \pm i, 0$

Example: 3.56 Find the critical points of the transformation $w = z + \frac{1}{z}$

Given $w = z + \frac{1}{z}$ ...(1)

Critical points occur at $\frac{dw}{dz} = 0$ and $\frac{dz}{dw} = 0$

Differentiation of (1) w. r. to z, we get

$$\Rightarrow \frac{dw}{dz} = 1 - \frac{1}{z^2} = \frac{z^2-1}{z^2}$$

Case $(i)\ \frac{dw}{dz} = 0$

$$\Rightarrow \frac{z^2-1}{z^3} = 0 \Rightarrow z^2 - 1 = 0 \Rightarrow z = \pm 1$$

Case $(ii)\ \frac{dz}{dw} = 0$

$$\Rightarrow \frac{z^3}{z^2-1} = 0 \Rightarrow z^2 = 0 \Rightarrow z = 0$$

$\therefore$ The critical points are $0, \pm 1$.