Problems based on Analytic functions – necessary conditions Cauchy – Riemann equations

Example: 3.6 Show that the function $f(z) = xy + iy$ is continuous everywhere but not differentiable anywhere.

Solution:

Given $f(z) = xy + iy$

$(i.e.)\ u = xy, v = y$

x and y are continuous everywhere and consequently $u(x,y) = xy$ and $v(x,y) = y$ are continuous everywhere.

Thus $f(z)$ is continuous everywhere.

But

$u = xy$	$v = y$
$u_x = y$	$v_x = 0$
$u_y = x$	$v_y = 1$
$u_x \neq v_y$	$u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not differentiable anywhere though it is continuous everywhere .

Example: 3.7 Show that the function $f(z) = \bar{z}$ is nowhere differentiable. [A.U N/D 2012]

Solution:

Given $f(z) = \bar{z} = x - iy$

i.e.,

$u = x$	$v = -y$
$\frac{\partial u}{\partial x} = 1$	$\frac{\partial v}{\partial x} = 0$
$\frac{\partial u}{\partial y} = 0$	$\frac{\partial v}{\partial y} = -1$

$\therefore u_x \neq v_y$

C−R equations are not satisfied anywhere.

Hence, $f(z) = \bar{z}$ is not differentiable anywhere (or) nowhere differentiable.

Example: 3.8 Show that $f(z) = |z|^2$ is differentiable at $z = 0$ but not analytic at $z = 0$.

Solution:

Let $z = x + iy$

$\bar{z} = x - iy$

$|z|^2 = z\bar{z} = x^2 + y^2$

$(i.e.)\ f(z) = |z|^2 = (x^2 + y^2) + i0$

$u = x^2 + y^2$	v= 0
$u_x = 2x$ $u_y = 2y$	$v_x = 0$ $v_y = 0$

So, the C−R equations $u_x = v_y$ and $u_y = -v_x$ are not satisfied everywhere except at $z = 0$.

So, f(z) may be differentiable only at $z = 0$.

Now, $u_x = 2x$, $u_y = 2y$, $v_x = 0$ and $v_y = 0$ are continuous everywhere and in particular at (0,0).

Hence, the sufficient conditions for differentiability are satisfied by $f(z)$ at $z = 0$.

So, $f(z)$ is differentiable at $z = 0$ only and is not analytic there.

Inverse function

Let $w = f(z)$ be a function of z and $z = F(w)$ be its inverse function.

Then the function $w = f(z)$ will cease to be analytic at $\frac{dz}{dw} = 0$ and $z = F(w)$ will be so, at point where $\frac{dw}{dz} = 0$.

Example: 3.9 Show that $f(z) = \log z$ analytic everywhere except at the origin and find its derivatives.

Solution:

Let $z = re^{i\theta}$

$f(z) = \log z$

$= \log(re^{i\theta}) = \log r + \log(e^{i\theta}) = \log r + i\theta$

But, at the origin, $r = 0$. Thus, at the origin,

$f(z) = \log 0 + i\theta = -\infty + i\theta$

Note : $e^{-\infty} = 0$ $\log e^{-\infty} = \log 0\,; -\infty = \log 0$

So, $f(z)$is not defined at the origin and hence is not differentiable there.

At points other than the origin, we have

$u(r,\theta) = \log r$	$v(r,\theta) = \theta$
$u_r = \frac{1}{r}$ $u_\theta = 0$	$v_r = 0$ $v_\theta = 1$

So, $\log z$ satisfies the C−R equations.

Further $\frac{1}{r}$ is not continuous at $z = 0$.

So, $u_r, u_\theta, v_r, v_\theta$ are continuous everywhere except at $z = 0$. Thus log z satisfies all the sufficient conditions for the existence of the derivative except at the origin. The derivative is

$$f'(z) = \frac{u_r + iv_r}{e^{i\theta}} = \frac{\left(\frac{1}{r}\right) + i(0)}{e^{i\theta}} = \frac{1}{re^{i\theta}} = \frac{1}{z}$$

Note: $f(z) = u + iv \Rightarrow f(re^{i\theta}) = u + iv$

Differentiate w.r.to 'r', we get

$$(i.e.)\ e^{i\theta} f'(re^{i\theta}) = \frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}$$

Example: 3.10 Check whether $w = \bar{z}$ is analytics everywhere. [Anna, Nov 2001] [A.U M/J 2014]

Solution:

Let $w = f(z) = \bar{z}$

$u + iv = x - iy$

$u = x$	$v = -y$
$u_x = 1$ $u_y = 0$	$v_x = 0$ $v_y = -1$

$u_x \neq v_y$ at any point p(x,y)

Hence, C−R equations are not satisfied.

∴The function $f(z)$ is nowhere analytic.

Example: 3.11 Test the analyticity of the function $w = \sin z$.

Solution:

Let $w = f(z) = sinz$

$u + iv = \sin(x + iy)$

$u + iv = \sin x \cos iy + \cos x \sin iy$

$u + iv = \sin x \cosh y + i \cos x \sin hy$

Equating real and imaginary parts, we get

$u = \sin x \cosh y$	$v = \cos x \sinh y$
$u_x = \cos x \cosh y$ $u_y = \sin x \sinh y$	$v_x = -\sin x \sinh y$ $v_y = \cos x \cosh y$

$\therefore u_x = v_y$ and $u_y = -v_x$

C −R equations are satisfied.

Also the four partial derivatives are continuous.

Hence, the function is analytic.

Example: 3.12 Determine whether the function $2xy + i(x^2 - y^2)$ is analytic or not. [Anna, May 2001]

Solution:

Let $f(z) = 2xy + i(x^2 - y^2)$

$(i.e.)$

$u = 2xy$	$v = x^2 - y^2$
$\frac{\partial u}{\partial x} = 2y$ $\frac{\partial u}{\partial y} = 2x$	$\frac{\partial v}{\partial x} = 2x$ $\frac{\partial v}{\partial y} = -2y$

$u_x \neq v_y$ and $u_y \neq -v_x$

C−R equations are not satisfied.

Hence, $f(z)$ is not an analytic function.

Example: 3.13 Prove that $f(z) = \cosh z$ is an analytic function and find its derivative.

Solution:

Given $f(z) = \cosh z = \cos(iz) = \cos[i(x+iy)]$

$= \cos(ix - y) = \cos ix \cos y + \sin(ix) \sin y$

$u + iv = \cosh x \cos y + i \sinh x \sin y$

$u = \cosh x \cos y$	$v = \sinh x \sin y$
$u_x = \sinh x \cos y$	$v_x = \cosh x \sin y$
$u_y = -\cosh x \sin y$	$v_y = \sinh x \cos y$

$\therefore u_x, u_y, v_x$ and v_y exist and are continuous.

$u_x = v_y$ and $u_y = -v_x$

C−R equations are satisfied.

$\therefore f(z)$ is analytic everywhere.

Now, $f'(z) = u_x + iv_x$

$= \sinh x \cos y + i \cosh x \sin y$

$= \sinh(x+iy) = \sinh z$

Test whether the function $\mathbf{f(z) = \frac{1}{2}\log(x^2 + y^2 + \tan^{-1}\left(\frac{y}{x}\right)}$ is analytic or not.

Solution:

Given $f(z) = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$(i.e.) u + iv = \frac{1}{2}\log(x^2 + y^2 + i\tan^{-1}\left(\frac{y}{x}\right)$

$u = \frac{1}{2}\log(x^2 + y^2)$	$v = \tan^{-1}\left(\frac{y}{x}\right)$
$u_x = \frac{1}{2}\frac{1}{x^2+y^2}(2x) = \frac{x}{x^2+y^2}$ $u_y = \frac{1}{2}\frac{1}{x^2+y^2}(2y) = \frac{y}{x^2+y^2}$	$v_x = \frac{1}{1+\frac{y^2}{x^2}}\left[-\frac{y}{x^2}\right] = \frac{-y}{x^2+y^2}$ $v_y = \frac{1}{1+\frac{y^2}{x^2}}\left[\frac{1}{x}\right] = \frac{x}{x^2+y^2}$

Here, $u_x = v_y$ and $u_y = -v_x$

$\Rightarrow$ C−R equations are satisfied

$\Rightarrow f(z)$ is analytic.