3.3 PROPERTIES – HARMONIC CONJUGATES

3.3 (a) Laplace equation

$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$ is known as Laplace equation in two dimensions.

3.3 (b) Laplacian Operator

$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is called the Laplacian operator and is denoted by ∇^2.

Note: (i) $\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0$ is known as Laplace equation in three dimensions.

Note: (ii) The Laplace equation in polar coordinates is defined as

$$\frac{\partial^2 \varphi}{\partial r^2} + \frac{1}{r}\frac{\partial \varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \varphi}{\partial \theta^2} = 0$$

3.3 (c) Harmonic function (or) [Potential function]

A real function of two real variables *x* and *y* that possesses continuous second order partial derivatives and that satisfies Laplace equation is called a harmonic function.

Note: A harmonic function is also known as a potential function.

Problems based on properties

Theorem: 1 If $f(z) = u + iv$ is a regular function of z in a domain D, then $\nabla^2|f(z)|^2 = 4|f'(z)|^2$

Solution:

Given $f(z) = u + iv$

$$\Rightarrow |f(z)| = \sqrt{u^2 + v^2}$$

$$\Rightarrow |f(z)|^2 = u^2 + v^2$$

$$\Rightarrow \nabla^2|f(z)|^2 = \nabla^2(u^2 + v^2)$$

$$= \nabla^2(u^2) + \nabla^2(v^2) \qquad \ldots(1)$$

$$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)u^2 += \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} \qquad \ldots(2)$$

$$\frac{\partial^2}{\partial x^2}(u^2) = \frac{\partial}{\partial x}\left[2u\frac{\partial u}{\partial x}\right] = 2\left[u\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\right] = 2u\frac{\partial^2 u}{\partial x^2} + 2\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

(2)$\Rightarrow$ $\nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \qquad [\because u \text{ is harmonic}]$$

$$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$$

Similarly, $\nabla^2(v^2) = 2v_x^2 + 2v_y^2$

(1)$\Rightarrow$ $\nabla^2|f(z)|^2 = 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$

$$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2] \qquad [\because u_x = v_y; u_y = -v_x]$$

$$= 4[u_x^2 + v_x^2]$$

$$(i.e.)\nabla^2|f(z)|^2 = 4|f'(z)|^2$$

> **Note :** $f(z) = u + iv$; $f'(z) = u_x + iv_x$;
>
> (or) $f'(z) = v_y + iu_y$; $|f'(z)| = \sqrt{u_x^2 + v_x^2}$; $|f'(z)|^2 = u_x^2 + v_x^2$

Theorem: 2 If $f(z) = u + iv$ is a regular function of z in a domain D, then $\nabla^2 \log |f(z)| = 0$ if $f(z)\, f'(z) \neq 0$ in D. i.e., $\log |f(z)|$ is harmonic in D. [A.U A/M 2017 R-13]

Solution:

Given $f(z) = u + iv$

$$|f(z)| = \sqrt{u^2 + v^2}$$

$$\log |f(z)| = \frac{1}{2} \log (u^2 + v^2)$$

$$\nabla^2 \log|f(z)| = \frac{1}{2} \nabla^2 \log (u^2 + v^2) = \frac{1}{2}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\log(u^2 + v^2)$$

$$= \frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2 + v^2)] \qquad \ldots(1)$$

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}[log(u^2 + v^2)] = \frac{1}{2}\frac{\partial^2}{\partial x}\left[\frac{1}{u^2+v^2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x}\right)\right] = \frac{\partial}{\partial x}\left[\frac{uu_x + vv_x}{u^2+v^2}\right]$$

$$= \frac{(u^2+v^2)[uu_{xx} + u_x u_x + vv_{xx} + v_x v_x] - (uu_x + vv_x)(2uu_x + 2vv_x)}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[uu_{xx} + vv_{xx} + u_x^2 + v_x^2] - 2(uu_x + vv_x)^2}{(u^2+v^2)^2}$$

Similarly, $\frac{1}{2}\frac{\partial^2}{\partial y^2}[log(u^2+v^2)] = \frac{(u^2+v^2)[uu_{yy}+vv_{yy}+u_y^2+v_y^2]-2(uu_y+vv_y)^2}{(u^2+v^2)^2}$

$$(1)\Rightarrow \nabla^2\log|f(z)| = \frac{(u^2+v^2)[u(u_{xx}+u_{yy})+v(v_{xx}+v_{yy})+(u_x^2+u_y^2)+(v_x^2+v_y^2)]-2[uu_x+vv_x]^2-2[uu_y+vv_y]^2}{(u^2+v^2)^2}$$

$$= \frac{(u^2+v^2)[u(0)+(u_x^2+v_x^2)+u_y^2+v_y^2)--2[u^2u_x^2+v^2v_x^2+2uv\,u_xv_x+u^2\,u_y^2+v^2v_y^2+2uv\,u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because u_{xx}+u_{yy}=0, v_{xx}+vv_{yy}=0]$

$$= \frac{(u^2+v^2)[|f'(z)^2|+|f'(z)^2|-2[u^2(u_x^2+u_y^2)+v^2(v_x^2+v_y^2)+2uv(u_x\,v_x+u_y\,v_y]}{(u^2+v^2)^2}$$

$[\because f'(z) = u+iv, |f'(z)| = u_x + iv_x$ (or) $f'(z) = v_y - iu_y, |f'(z)|^2 = u_x^2 + v_x^2$

(or) $|f'(z)|^2 = u_y^2 + v_y^2$

$$= \frac{2(u^2+v^2)[|f'(z)^2|-2[u^2|f'(z)|^2+v^2|f'(z)^2|+2uv(0)]}{(u^2+v^2)^2}$$

$[\because u_x = v_y,\ u_y = -v_x]$

$\Rightarrow u_x\,v_x + u_yv_y = 0$

$\Rightarrow u_x^2 + u_y^2 = u_x^2 + v_x^2 = |f'(z)|^2$

$\Rightarrow v_x^2 + v_y^2 = u_y^2 + v_y^2 = |f'(z)|^2$

$$= \frac{2(u^2+v^2)|f(z)|^2-2(u^2+v^2)|f'(z)|^2}{(u^2+v^2)^2}$$

$(i.e.)\ \nabla^2\log|f(z)| = 0$

Problems based on properties

Theorem: 1 If $f(z) = u+iv$ is a regular function of z in a domain D, then $\nabla^2|f(z)|^2 = 4|f'(z)|^2$

Solution:

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$= \nabla^2(u^2) + \nabla^2(v^2) \quad \ldots(1)$

$\nabla^2(u^2) = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)u^2 += \frac{\partial^2(u^2)}{\partial x^2} + \frac{\partial^2(u^2)}{\partial y^2} \quad \ldots(2)$

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Similarly, $\frac{\partial^2}{\partial y^2}(u^2) = 2u\frac{\partial^2 u}{\partial y^2} + 2\left(\frac{\partial u}{\partial y}\right)^2$

$(2)\Rightarrow \nabla^2(u^2) = 2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right) + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right]$

$= 0 + 2\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right] \quad [\therefore u$ is harmonic]

$\nabla^2(u^2) = 2u_x^2 + 2u_y^2$

Similarly, $\nabla^2(v^2) = 2v_x^2 + 2v_y^2$

$(1)\Rightarrow \nabla^2|f(z)|^2 = 2[u_x^2 + u_y^2 + v_x^2 + v_y^2]$

$= 2[u_x^2 + (-v_x)^2 + v_x^2 + u_x^2] \quad [\because u_x = v_y; u_y = -v_x]$