Unit-III

Complex Integration

# ANALYTIC FUNCTIONS

## 3.1 INTRODUCTION

The theory of functions of a complex variable is the most important in solving a large number of Engineering and Science problems. Many complicated intergrals of real function are solved with the help of a complex variable.

## 3.1 (a) Complex Variable

$x + iy$ is a complex variable and it is denoted by $z$.

$(i.e.)z = x + iy \text{ where } i = \sqrt{-1}$

## 3.1 (b) Function of a complex Variable

If $z = x + iy$ and $w = u + iv$ are two complex variables, and if for each value of $z$ in a given region R of complex plane there corresponds one or more values of $w$ is said to be a function $z$ and is denoted by $w = f(z) = f(x + iy) = u(x, y) + iv(x, y)$where $u(x, y)$and $v(x, y)$are real functions of the real variables $x$ and $y$.

**Note:**

### (i) single-valued function

If for each value of $z$ in R there is correspondingly only one value of $w$, then $w$ is called a single valued function of $z$.

**Example:** $w = z^2, w = \frac{1}{z}$

| $w = z^2$ | | | | | $w = \frac{1}{z}$ | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $z$ | 1 | 2 | $-2$ | 3 | $z$ | 1 | 2 | $-2$ | 3 |
| $w$ | 1 | 4 | 4 | 9 | $w$ | 1 | $\frac{1}{2}$ | $\frac{1}{-2}$ | $\frac{1}{3}$ |

## (ii) Multiple – valued function

If there is more than one value of $w$ corresponding to a given value of $z$ then $w$ is called multiple – valued function.

**Example:** $w = z^{1/2}$

| $w = z^{1/2}$ | | | |
|---|---|---|---|
| $z$ | 4 | 9 | 1 |
| $w$ | $-2,2$ | $3,-3$ | $1,-1$ |

**(iii)** The distance between two points $z$and $z_o$ is $|z - z_o|$

**(iv)**The circle C of radius $\delta$ with centre at the point $z_o$can be represented by $|z - z_o| = \delta$.

**(v)** $|z - z_o| < \delta$ represents the interior of the circle excluding its circumference.

**(vi)** $|z - z_o| \leq \delta$ represents the interior of the circle including its circumference.

**(vii)** $|z - z_o| > \delta$ represents the exterior of the circle.

**(viii)** A circle of radius 1 with centre at origin can be represented by $|z| = 1$

## 3.1 (c) Neighbourhood of a point $z_o$

Neighbourhood of a point $z_o$, we mean a sufficiently small circular region [excluding the points on the boundary] with centre at $z_o$.

$$(i.e.)\ |z - z_o| < \delta$$

Here, $\delta$ is an arbitrary small positive number.

## 3.1 (d) Limit of a Function

Let $f(z)$ be a single valued function defined at all points in some neighbourhood of point $z_o$.

Then the limit of $f(z)$ as z approaches $z_o is\ w_o$.

$$(i.e.)\ \lim_{z \to z_o} f(z) = w_o$$

## 3.1 (e) Continuity

If $f(z)$ is said to continuous at $z = z_o$ then

$$\lim_{z \to z_o} f(z) = f(z_o)$$

If two functions are continuous at a point their sum, difference and product are also continuous at that point, their quotient is also continuous at any such point $[dr \neq 0]$

**Example: 3.1 State the basic difference between the limit of a function of a real variable and that of a complex variable. [A.U M/J 2012]**

**Solution:**

In real variable, $x \to x_0$ implies that $x$ approaches $x_0$ along the X-axis (or) a line parallel to the X-axis.

In complex variables, $z \to z_0$ implies that z approaches $z_0$ along any path joining the points $z \, and \, z_0$ that lie in the z-plane.

## 3.1 (f) Differentiability at a point

A function $f(z)$ is said to be differentiable at a point, $z = z_0$ if the limit

$$f(z_0) = \underset{\Delta z \to 0}{\text{Lt}} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \text{ exists.}$$

This limit is called the derivative of $f(z)$at the point $z = z_0$

If $f(z)$ is differentiable at $z_0$, then $f(z)$is continuous at $z_0$. This is the necessary condition for differentiability.

**Example: 3.2 If $f(z)$ is differentiable at $z_0$, then show that it is continuous at that point.**

**Solution:**

As $f(z)$ is differentiable at $z_0$, both $f(z_0)$ and $f'(z_0)$exist finitely.

Now, $\lim\limits_{z \to z_o} |f(z) - f(z_o)| = \lim\limits_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0}(z - z_0)$

$$= \lim_{z \to z_o} \frac{f(z) - f(z_o)}{z - z_0} \lim_{z \to z_o} (z - z_0)$$

$$= f'(z_0).0 = 0$$

Hence, $\lim\limits_{z \to z_o} f(z) = \lim\limits_{z \to z_o} f(z_o) = f(z_o)$

As $f(z_0)$ is a constant.

This is exactly the statement of continuity of $f(z)$ at $z_0$.

**Example: 3.3 Give an example to show that continuity of a function at a point does not imply the existence of derivative at that point.**

**Solution:**

Consider the function $w = |z|^2 = x^2 + y^2$

This function is continuous at every point in the plane, being a continuous function of two real variables. However, this is not differentiable at any point other than origin.

**Example: 3.4 Show that the function f(z) is discontinuous at $z = 0$, given that $f(z) = \frac{2xy^2}{x^2+3y^4}$, when $z \neq 0$ and $f(0) = 0$.**

**Solution:**

Given $f(z) = \frac{2xy^2}{x^2+3y^4}$,

Consider $\lim_{z \to z_0} [f(z)] = \lim_{\substack{y=mx \\ x\to 0}} [f(z)] = \lim_{x\to 0} \frac{2x(mx)^2}{x^2+3(mx)^4} = \lim_{x\to 0} \left[\frac{2m^2x}{1+3m^4x^2}\right] = 0$

$$\lim_{\substack{y^2=x \\ x\to 0}} [f(z)] = \lim_{x\to 0} \frac{2x^2}{x^2+3x^2} = \lim_{x\to 0} \frac{2x^2}{4x^2} = \frac{2}{4} = \frac{1}{2} \neq 0$$

$\therefore f(z)$ is discontinuous

**Example: 3.5 Show that the function $f(z)$ is discontinuous at the origin $(z = 0)$, given that**

$f(z) = \frac{xy(x-2y)}{x^3+y^3}$, **when $z \neq 0$**

$= 0$ **, when $z = 0$**

**Solution:**

Consider $\lim_{z \to z_0} [f(z)] = \lim_{\substack{y=mx \\ x\to 0}} [f(z)] = \lim_{x\to 0} \frac{x(mx)(x-2(mx))}{x^3+(mx)^3}$

$$= \lim_{x\to 0} \frac{m(1-2m)x^3}{(1+m^3)x^3} = \frac{m(1-2m)}{1+m^3}$$

Thus $\lim_{z\to 0} f(z)$ depends on the value of m and hence does not take a unique value.

$\therefore \lim_{z\to 0} f(z)$ does not exist.

$\therefore f(z)$ is discontinuous at the origin.

**3.2 (i) The necessary condition for $f = (z)$ to be analytic. [Cauchy – Riemann Equations]**

The necessary conditions for a complex function $f = (z) = u(x,y) + iv(x,y)$ to be analytic in a region R are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $and$ $\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$ $i.e., u_x = v_y$ $and$ $v_x = -u_y$

**(ii) Sufficient conditions for $f(z)$ to be analytic.**

If the partial derivatives $u_x, u_y, v_x$ and $v_y$ are all continuous in D and $u_x = v_y$ and $u_y = -v_x$, then the function $f(z)$ is analytic in a domain D.

**(ii) Polar form of C-R equations**

In Cartesian co-ordinates any point z is $z = x + iy$.

In polar co-ordinates, $z = re^{i\theta}$ where r is the modulus and $\theta$ is the argument.