UNIT - II

VECTOR CALCULUS

INTRODUCTION

In this chapter we study the basics of vector calculus with the help of a standard vector differential operator. Also we introduce concepts like gradient of a scalar valued function, divergence and curl of a vector valued function, discuss briefly the properties of these concepts and study the applications of the results to the evaluation of line and surface integrals in terms of multiple integrals.

2.1 GRADIENT – DIRECTIONAL DERIVATIVE

Vector differential operator

The vector differential operator ∇ (read as Del) is denoted by $\nabla = \vec{\imath}\frac{\partial}{\partial x} + \vec{\jmath}\frac{\partial}{\partial y} + \vec{k}\frac{\partial}{\partial z}$ where $\vec{\imath}, \vec{\jmath}, \vec{k}$ are unit vectors along the three rectangular axes OX, OY and OZ.

It is also called Hamiltonian operator and it is neither a vector nor a scalar, but it behaves like a vector.

The gradient of a scalar function

If$\varphi(x, y, z)$ is a scalar point function continuously differentiable in a given region of space, then the gradient of φ is defined as $\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$

It is also denoted as Grad φ.

Note

(i) $\nabla\varphi$ is a vector quantity.

(ii) $\nabla\varphi = 0$ if φ is constant.

(iii) $\nabla(\varphi_1\varphi_2) = \varphi_1\nabla\varphi_2 + \varphi_2\nabla\varphi_1$

(iv) $\nabla\left(\frac{\varphi_1}{\varphi_2}\right) = \frac{\varphi_2\nabla\varphi_1 - \varphi_1\nabla\varphi_2}{{\varphi^2}_2}$ if $\varphi_2 \neq 0$

(v) $\nabla(\varphi \pm \chi) = \nabla\varphi \pm \nabla\chi$

Problems based on Gradient

Example: 2.1 Find the gradient of φ where φ is $3x^2y - y^3z^2$ at $(1,-2,1)$.

Solution:

Given $\varphi = 3x^2y - y^3z^2$

$$\text{Grad } \varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 6xy$, $\frac{\partial\varphi}{\partial y} = 3x^2 - 3y^2z^2$, $\frac{\partial\varphi}{\partial z} = -2y^3z$

$$\therefore \text{grad } \varphi = \vec{\imath}\,6xy + \vec{\jmath}(3x^2 - 3y^2z^2) - \vec{k}2y^3z$$

$$\therefore (grad\ \varphi)_{(1,-2,\ 1)} = -12\vec{\imath} - 9\vec{\jmath} + 16\vec{k}$$

Example: 2.2 If $\varphi = \log(x^2 + y^2 + z^2)$ then find $\nabla\varphi$.

Solution:

Given $\varphi = \log(x^2 + y^2 + z^2)$

$$\nabla\varphi = \vec{\imath}\frac{\partial\varphi}{\partial x} + \vec{\jmath}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{\imath}\left(\frac{2x}{x^2+y^2+z^2}\right) + \vec{\jmath}\left(\frac{2y}{x^2+y^2+z^2}\right) + \vec{k}\left(\frac{2z}{x^2+y^2+z^2}\right)$$

$$= \frac{2}{x^2+y^2+z^2}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = \frac{2}{r^2}\vec{r}$$

Example: 2.3 Find $\nabla(r), \nabla\left(\frac{1}{r}\right), \nabla(\log r)$ where $r = |\vec{r}|$ and $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$.

Solution:

Given $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

$$\Rightarrow |\vec{r}| = r = \sqrt{x^2 + y^2 + z^2}$$

$$\Rightarrow r^2 = x^2 + y^2 + z^2$$

$$2r\frac{\partial r}{\partial x} = 2x, \quad 2r\frac{\partial r}{\partial y} = 2y, \quad 2r\frac{\partial r}{\partial z} = 2z$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

(i) $\nabla(r) = \vec{\imath}\frac{\partial r}{\partial x} + \vec{\jmath}\frac{\partial r}{\partial y} + \vec{k}\frac{\partial r}{\partial z}$

$$= \vec{\imath}\frac{x}{r} + \vec{\jmath}\frac{y}{r} + \vec{k}\frac{z}{r}$$

$$= \frac{1}{r}\left(x\vec{\imath} + y\vec{\jmath} + z\vec{k}\right) = \frac{1}{r}\vec{r}$$

(ii) $\nabla\left(\frac{1}{r}\right) = \vec{i}\frac{\partial\left(\frac{1}{r}\right)}{\partial x} + \vec{j}\frac{\partial\left(\frac{1}{r}\right)}{\partial y} + \vec{k}\frac{\partial\left(\frac{1}{r}\right)}{\partial z}$

$$= \vec{i}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial x} + \vec{j}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial y} + \vec{k}\left(\frac{-1}{r^2}\right)\frac{\partial r}{\partial z}$$

$$= \left(-\frac{1}{r^2}\right)\left[\vec{i}\frac{x}{r} + \vec{j}\frac{y}{r} + \vec{k}\frac{z}{r}\right]$$

$$= -\frac{1}{r^3}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = -\frac{1}{r^3}\vec{r}$$

(iii) $\nabla(\log r) = \sum \vec{i}\frac{\partial(\log r)}{\partial x}$

$$= \sum \vec{i}\frac{1}{r}\frac{\partial r}{\partial x}$$

$$= \sum \vec{i}\frac{1}{r}\frac{x}{r}$$

$$= \sum \vec{i}\frac{x}{r^2}$$

$$= \frac{1}{r^2}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = \frac{1}{r^2}\vec{r}$$

Example: 2.4 Prove that $\nabla(r^n) = nr^{n-2}\,\vec{r}$

Solution:

Given $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$

$$\nabla(r^n) = \vec{i}\frac{\partial r^n}{\partial x} + \vec{j}\frac{\partial r^n}{\partial y} + \vec{k}\frac{\partial r^n}{\partial z}$$

$$= \vec{i}\,nr^{n-1}\frac{\partial r}{\partial x} + \vec{j}\,nr^{n-1}\frac{\partial r}{\partial y} + \vec{k}\,nr^{n-1}\frac{\partial r}{\partial z}$$

$$= nr^{n-1}\left[\vec{i}\left(\frac{x}{r}\right) + \vec{j}\left(\frac{y}{r}\right) + \vec{k}\left(\frac{z}{r}\right)\right]$$

$$= \frac{nr^{n-1}}{r}\left(x\vec{i} + y\vec{j} + z\vec{k}\right) = nr^{n-2}\vec{r}$$

Example: 2.5 Find $|\nabla\varphi|$ *if* $\varphi = 2xz^4 - x^2y$ *at* $(2, -2, -1)$

Solution:

Given $\varphi = 2xz^4 - x^2y$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

Now $\frac{\partial\varphi}{\partial x} = 2z^4 - 2xy,\quad \frac{\partial\varphi}{\partial y} = -x^2,\quad \frac{\partial\varphi}{\partial z} = 8xz^3$

$$\therefore \nabla\varphi = \vec{i}\,(2z^4 - 2xy) + \vec{j}(-x^2) + \vec{k}(8xz^3)$$

$$\therefore (\nabla\varphi)_{(2,-2,-1)} = 10\vec{i} - 4\vec{j} - 16\vec{k}$$

$$|\nabla\varphi| = \sqrt{100 + 16 + 256} = \sqrt{372}$$

Directional Derivative (D.D) of a scalar point function

The derivative of a point function (scalar or vector) in a particular direction is called its directional derivative along the direction.

The directional derivative of a scalar function φ in a given direction $\vec{a}$ is the rate of change of φ in that direction. It is given by the component of $\nabla\varphi$ in the direction of $\vec{a}$.

The directional derivative of a scalar point function in the direction of $\vec{a}$ is given by

$$\mathbf{D.D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

The maximum directional derivative is $|\nabla\varphi|$ or $|\text{grad } \varphi|$.

Problems based on Directional Derivative

Example: 2.6 Find the directional derivative of $\varphi = 4xz^2 + x^2yz$ at $(1, -2, 1)$ in the direction of $2\vec{i} - \vec{j} - 2\vec{k}$.

Solution:

Given $\varphi = 4xz^2 + x^2yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xyz + 4z^2) + \vec{j}\,(x^2z) + \vec{k}\,(x^2y + 8xz)$$

$$\therefore (\nabla\,\varphi)_{(1,-2,-1)} = 8\vec{i} - \vec{j} - 10\vec{k}$$

Given $\vec{a} = 2\vec{i} - \vec{j} - 2\vec{k}$

$$|\vec{a}| = \sqrt{4+1+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \left(8\vec{i} - \vec{j} - 10\vec{k}\right) \cdot \frac{(2\vec{i} - \vec{j} - 2\vec{k})}{3}$$

$$= \frac{1}{3}\,(16 + 1 + 20) = \frac{37}{3}$$

Example: 2.7 Find the directional derivative of $\varphi(x,y,z) = xy^2 + yz^3$ at the point P$(2,-1,1)$ in the direction of PQ where Q is the point $(3,1,3)$

Solution:

Given $\varphi = xy^2 + yz^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2) + \vec{j}\,(2xy + z^3) + \vec{k}\,(3yz^2)$$

$$\therefore (\nabla\varphi)_{(2,-1,\ 1)} = \vec{i} - 3\vec{j} - 3\vec{k}$$

Given $\vec{a} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP}$

$$= (3\vec{i} + \vec{j} + 3\vec{k}) - (2i - \vec{j} + \vec{k})$$

$$= \vec{i} + 2\vec{j} + 2\vec{k}$$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi\cdot\vec{a}}{|\vec{a}|}$$

$$= \frac{(\vec{i}-3\vec{j}-3\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{3}$$

$$= \frac{1}{3}(1-6-6) = -\frac{11}{3}$$

Example: 2.8 In what direction from $(-1,1,2)$ is the directional derivative of $\varphi = xy^2 z^3$ a maximum? Find also the magnitude of this maximum.

Solution:

Given $\varphi = xy^2 z^3$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y^2z^3) + \vec{j}\,(2xy\,z^3) + \vec{k}\,(3xy^2z^2)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 2)} = 8\vec{i} - 16\vec{j} - 12\vec{k}$$

The maximum directional derivative occurs in the direction of $\nabla\varphi = 8\vec{i} - 16\vec{j} - 12\vec{k}$.

$\therefore$ The magnitude of this maximum directional derivative

$$|\nabla\varphi| = \sqrt{64+256+144} = \sqrt{464}$$

Example: 2.9 Find the directional derivative of the scalar function $\varphi = xyz$ in the direction of the outer normal to the surface $z = xy$ at the point$(3, 1, 3)$.

Solution:

Given $\varphi = xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(yz) + \vec{j}\,(xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\varphi)_{(3,\ 1,\ 3)} = 3\vec{i} + 9\vec{j} + 3\vec{k}$$

Given surface is $z = xy \Rightarrow z - xy = 0$

$$\nabla\chi = \vec{i}\frac{\partial\chi}{\partial x} + \vec{j}\frac{\partial\chi}{\partial y} + \vec{k}\frac{\partial\chi}{\partial z}$$

$$= \vec{i}\,(-y) + \vec{j}\,(-x) + \vec{k}\,(1)$$

Let $\vec{a} = \nabla\chi_{(3,1,3)} = -\vec{i} - 3\vec{j} + \vec{k}$

$$\Rightarrow |\vec{a}| = \sqrt{1+9+1} = \sqrt{11}$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(3\vec{i}+9\vec{j}+3\vec{k})\cdot(-\vec{i}-3\vec{j}+\vec{k})}{\sqrt{11}}$$

$$= \frac{1}{\sqrt{11}}(-3-27+3) = -\frac{27}{\sqrt{11}}$$

Example: 2.10 Find the directional derivative of $\varphi = xy + yz + zx$ at $(1, 2, 0)$ in the direction of $\vec{i} + 2\vec{j} + 2\vec{k}$. Find also its maximum value.

Solution:

Given $\varphi = xy + yz + zx$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y+z) + \vec{j}\,(x+z) + \vec{k}\,(y+x)$$

$$\therefore (\nabla\varphi)_{(1,\ 2,\ 0)} = 2\vec{i} + \vec{j} + 3\vec{k}$$

Given $\vec{a} = \vec{i} + 2\vec{j} + 2\vec{k}$

$$|\vec{a}| = \sqrt{1+4+4} = 3$$

$$\text{D. D} = \frac{\nabla\varphi \cdot \vec{a}}{|\vec{a}|}$$

$$= \frac{(2\vec{i}+\vec{j}+3\vec{k})\cdot(\vec{i}+2\vec{j}+2\vec{k})}{3}$$

$$= \frac{1}{3}(2+2+6) = \frac{10}{3}$$

Maximum value is $|\nabla\varphi| = \sqrt{4+1+9} = \sqrt{14}$

Unit normal vector to the surface

If $\varphi(x, y, z)$ be a scalar function, then $\varphi(x, y, z) = c$ represents a surface and the unit normal vector to the surface φ is given by $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|}$

Normal Derivative $= |\nabla\varphi|$

Problems based on unit normal vector

Example: 2.11 Find the unit normal to the surface $x^2 + y^2 = z$ at the point $(1, -2, 5)$.

Solution:

Given $\varphi = x^2 + y^2 - z$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 5)} = 2\vec{i} - 4\vec{j} - \vec{k}$$

$$|\nabla\varphi| = \sqrt{4 + 16 + 1} = \sqrt{21}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{2\vec{i} - 4\vec{j} - \vec{k}}{\sqrt{21}}$

Example: 2.12 Find the unit normal to the surface $x^2 + xy + y^2 + xyz$ at the point $(1, -2, 1)$.

Solution:

Given $\varphi = x^2 + xy + y^2 + xyz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x + y + yz) + \vec{j}\,(x + 2y + xz) + \vec{k}\,(xy)$$

$$\therefore (\nabla\varphi)_{(1,-2,\ 1)} = -2\vec{i} - 2\vec{j} - 2\vec{k}$$

$$|\nabla\varphi| = \sqrt{4 + 4 + 4} = \sqrt{12} = 2\sqrt{3}$$

Unit normal $\hat{n} = \frac{\nabla\varphi}{|\nabla\varphi|} = \frac{-2\vec{i} - 2\vec{j} - 2\vec{k}}{2\sqrt{3}}$

$$= \frac{-1}{\sqrt{3}}\left(\vec{i} + \vec{j} + \vec{k}\right)$$

Example: 2.13 Find the normal derivative to the surface $x^2y + xz^2$ at the point $(-1, 1, 1)$.

Solution:

Given $\varphi = x^2y + xz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2xy + z^2) + \vec{j}\,(x^2) + \vec{k}\,(2xz)$$

$$\therefore (\nabla\varphi)_{(-1,\ 1,\ 1)} = -\vec{i} + \vec{j} - 2\vec{k}$$

Normal derivative $|\nabla\varphi| = \sqrt{1+1+4} = \sqrt{6}$

Example: 2.14 What is the greatest rate of increase of $\varphi = xyz^2$ at the point $(1, 0, 3)$.

Solution:

Given $\varphi = xyz^2$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(y\,z^2) + \vec{j}\,(xz^2) + \vec{k}\,(2xyz)$$

$$\therefore (\nabla\varphi)_{(1,\ 0,\ 3)} = 0\vec{i} + 9\vec{j} + 0\vec{k}$$

$\therefore$ Greatest rate of increase $|\nabla\varphi| = \sqrt{9^2} = 9$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$$

Angle between the surfaces

$$\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1||\nabla\varphi_2|}\right]$$

Example: 2.15 Find the angle between the surfaces $z = x^2 + y^2 - 3$ and $x^2 + y^2 + z^2 = 9$ at the point $(2, -1, 2)$.

Solution:

Given $\varphi = x^2 + y^2 - z - 3$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(-1)$$

$$\therefore (\nabla\varphi_1)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{16+4+1} = \sqrt{21}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(2y) + \vec{k}\,(2z)$$

$$\therefore (\nabla\varphi_2)_{(2,-1,\ 2)} = 4\vec{i} - 2\vec{j} + 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16+4+16} = \sqrt{36} = 6$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(4\vec{i}-2\vec{j}-\vec{k})\,(4\vec{i}-2\vec{j}+4\vec{k})}{\sqrt{21}(6)}$$

$$= \frac{16+4-4}{\sqrt{21}(6)}$$

$$= \frac{16}{\sqrt{21}(6)} = \frac{8}{3\sqrt{21}}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{8}{3\sqrt{21}}\right]$$

Example: 2.16 Find the angle between the normals to the surfaces $x^2 = yz$ at the point $(1, 1, 1)$ and $(2, 4, 1)$.

Solution:

Given $\varphi = x^2 - yz$

$$\nabla\varphi = \vec{i}\frac{\partial\varphi}{\partial x} + \vec{j}\frac{\partial\varphi}{\partial y} + \vec{k}\frac{\partial\varphi}{\partial z}$$

$$= \vec{i}\,(2x) + \vec{j}\,(-z) + \vec{k}\,(-y)$$

$$\therefore (\nabla\,\varphi_1)_{(1,\ 1,\ 1)} = 2\vec{i} - \vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{4 + 1 + 1} = \sqrt{6}$$

$$\therefore (\nabla\,\varphi_2)_{(2,\ 4,\ 1)} = 4\vec{i} - \vec{j} - 4\vec{k}$$

$$|\nabla\varphi_2| = \sqrt{16 + 1 + 16} = \sqrt{33}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\,|\nabla\varphi_2|}$

$$= \frac{(2\vec{i} - \vec{j} - \vec{k})\,(4\vec{i} - \vec{j} - 4\vec{k})}{\sqrt{6}\sqrt{33}}$$

$$= \frac{8+1+4}{\sqrt{6}\sqrt{33}}$$

$$= \frac{13}{\sqrt{2(3)}\sqrt{11(3)}} = \frac{13}{3\sqrt{22}}$$

$$\Rightarrow \theta = cos^{-1}\left[\frac{13}{3\sqrt{22}}\right]$$

Example: 2.17 Find the angle between the surfaces $x\log z = y^2 - 1$ and $x^2y = 2 - z$ at the point $(1, 1, 1)$.

Solution:

Given $\varphi_1 = y^2 - x\log z - 1$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\,(-\log z) + \vec{j}\,(2y) + \vec{k}\left(-\frac{x}{z}\right)$$

$$\therefore (\nabla\varphi_1)_{(1,\ 1,\ 1)} = 0\vec{i} + 2\vec{j} - \vec{k}$$

$$|\nabla\varphi_1| = \sqrt{0+4+1} = \sqrt{5}$$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(2xy) + \vec{j}\,(x^2) + \vec{k}\,(1)$$

$$\therefore (\nabla\varphi_2)_{(1,\ 1,\ 1)} = 2\vec{i} + \vec{j} + \vec{k}$$

$$|\nabla\varphi_2| = \sqrt{4+1+1} = \sqrt{6}$$

The angle between the surfaces is $\cos\theta = \frac{\nabla\varphi_1 \cdot \nabla\varphi_2}{|\nabla\varphi_1|\ |\nabla\varphi_2|}$

$$= \frac{(0\vec{i}+2\vec{j}-\vec{k})\cdot(2\vec{i}+\vec{j}+\vec{k})}{\sqrt{5}\sqrt{6}}$$

$$= \frac{0+2-1}{\sqrt{30}}$$

$$= \frac{1}{\sqrt{30}}$$

$$\Rightarrow \theta = \cos^{-1}\left[\frac{1}{\sqrt{30}}\right]$$

Problems based on orthogonal surfaces

Two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\varphi_2 = 0$

Example: 2.18 Find a and b such that the surfaces $ax^2 - byz = (a+2)x$ and $4x^2y + z^3 = 4$ cut orthogonally at $(1, -1, 2)$.

Solution:

Given $ax^2 - byz = (a+2)x$

Let $\varphi_1 = ax^2 - byz - (a+2)x$

$$\nabla\varphi_1 = \vec{i}\frac{\partial\varphi_1}{\partial x} + \vec{j}\frac{\partial\varphi_1}{\partial y} + \vec{k}\frac{\partial\varphi_1}{\partial z}$$

$$= \vec{i}\left(2ax - (a+2)\right) + \vec{j}\,(-bz) + \vec{k}\,(-by)$$

$\therefore (\nabla\,\varphi_1)_{(1,-1,\ 2)} = \vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)$

Let $\varphi_2 = 4x^2y + z^3 - 4$

$$\nabla\varphi_2 = \vec{i}\frac{\partial\varphi_2}{\partial x} + \vec{j}\frac{\partial\varphi_2}{\partial y} + \vec{k}\frac{\partial\varphi_2}{\partial z}$$

$$= \vec{i}\,(8xy) + \vec{j}\,(4x^2) + \vec{k}\,(3z^2)$$

$\therefore (\nabla\,\varphi_2)_{(1,-1,\ 2)} = -8\vec{i} + 4\vec{j} + 12\vec{k}$

Since the two surfaces are orthogonal if $\nabla\varphi_1 \cdot \nabla\,\varphi_2 = 0$

$\Rightarrow \left(\vec{i}(a-2) + \vec{j}\,(-2b) + \vec{k}(b)\right) \cdot \left(-8\vec{i} + 4\vec{j} + 12\vec{k}\,\right) = 0$

$\Rightarrow\ -8\,(a-2) - 8b + 12b = 0$

$\Rightarrow\ -8a + 16 - 8b + 12b = 0$

$\Rightarrow\ -8a + 16 + 4b = 0$

$\div$ by 4 $\Rightarrow\ -2a + 4 + b = 0$

$$\Rightarrow 2a - b - 4 = 0\ \ \dots\ (1)$$

To find a and b we need another equation in a and b.

The point $(1,-1,2)$ lies in $ax^2 - byz - (a+2)x = 0$

$\therefore a - b\,(-1)\,(2) - (a+2)\,(1) = 0$

$\Rightarrow a + 2b - a - 2 = 0$

$\Rightarrow 2b - 2 = 0$

$\Rightarrow b = 1$

Substitute $b = 1$ in (1) we get

$\Rightarrow 2a - 1 - 4 = 0$

$\Rightarrow 2a - 5 = 0$

$\Rightarrow a = \frac{5}{2}$

Exercise: 2.1

1. Find $\nabla\varphi$ if $\varphi = \frac{1}{2}\log(x^2 + y^2 + z^2)$ **Ans:** $\frac{\vec{r}}{r^2}$

2. Find the directional derivative of

   (i) $\varphi = 2xy + z^2$ at the point $(1, -1, 3)$ in the direction $\vec{\imath} + 2\vec{\jmath} + 2\vec{k}$. **Ans:** $\frac{14}{3}$

   (ii) $\varphi = xy^2 + yz^3$ at the point $(2, -1, 1)$ in the direction of PQ where Q is the point $(3, 1, 3)$. **Ans:** $\frac{-11}{3}$

3. Prove that the directional derivative of $\varphi = x^3y^2z$ at $(1, 2, 3)$is maximum along the direction $9\vec{\imath} + 3\vec{\jmath} + \vec{k}$. Also, find the maximum directional derivative. **Ans:** $4\sqrt{91}$

4. Find the unit tangent vector to the curve $\vec{r} = (t^2 + 1)\vec{\imath} + (4t - 3)\vec{\jmath} + (2t^2 - 65)\vec{k}$ at $t = 1$. **Ans:** $\frac{\vec{\imath}+2\vec{\jmath}-\vec{k}}{\sqrt{6}}$

5. Find a unit normal to the following surfaces at the specified points.

   (i) $x^2y + 2xz = 4$ at $(2, -2, 3)$ **Ans:** $\pm\frac{1}{3}(\vec{\imath} - 2\vec{\jmath} - 2\vec{k})$

   (ii) $x^2 + y^2 = z$ at $(1, -2, 5)$ **Ans:** $\frac{1}{\sqrt{21}}(2\vec{\imath} - 4\vec{\jmath} - \vec{k})$

   (ii) $xy^3z^2 = 4$ at $(-1, -1, 2)$ **Ans:** $\frac{1}{\sqrt{11}}(-\vec{\imath} - 3\vec{\jmath} + \vec{k})$

   (iv) $x^2 + y^2 = z$ at $(1, 1, 2)$ **Ans:** $\frac{1}{3}(2\vec{\imath} + 2\vec{\jmath} - \vec{k})$