2.5 STOKE'S THEOREM

Statement of Stoke's theorem

If S is an open surface bounded by a simple closed curve C if $\vec{F}$ is continuous having continuous partial derivatives in S and C, then

$$\int_c \vec{F} \cdot d\vec{r} = \iint_S curl\,\vec{F} \cdot \hat{n}\, ds$$

(or)

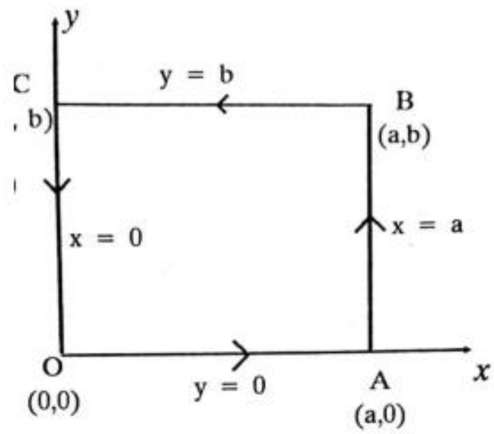
$$\int_c \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \hat{n}\, ds$$

$\hat{n}$ is the outward unit normal vector and C is traversed in the anti – clockwise direction.

Problems based on Stoke's theorem

Example: 2.72 Verify stokes theorem for a vector field defined by $\vec{F} = (x^2 - y^2)\vec{i} + 2xy\vec{j}$ in a rectangular region in the xoy plane bounded by the lines $x = 0, x = a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\, \vec{F} \cdot \hat{n}\, dS$

To evaluate: $\iint_S Curl\, \vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 - y^2)\vec{i} + 2\,x\,y\vec{j}$

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 - y^2 & 2xy & 0 \end{vmatrix}$$

$= \vec{i}\,(0) - \vec{j}\,(0 - 0) + \vec{k}[\,2y - (0 - 2y)]$

$= 4y\,\vec{k}$

Since the surface is a rectangle in the xy plane, $\hat{n} = \vec{k}$, $dS = dxdy$

Curl $\vec{F} \cdot \hat{n} = 4y\ \vec{k} \cdot \vec{k} = 4y$

Order of integration is $dxdy$

x varies from $x = 0$ to $x = a$

y varies from $y = 0$ to $y = b$

$$\Rightarrow \iint_S Curl\, \vec{F} \cdot \hat{n}\, dS = \int_0^b \int_0^a 4y\, dx\, dy$$

$$= \int_0^b 4y\, [x]_0^a dy$$

$$= \int_0^b 4ay dy$$

$$= \left[\frac{4ay^2}{2}\right]_0^b$$

$$= 2ab^2$$

$$\Rightarrow \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = 2ab^2 \quad \ldots(1)$$

Here the line integral over the simple closed curve C bounding the surface $OABCO$ consisting of the edges OA, AB ,BC and CO.

Curve	Equation	Limit
OA	$y = 0$	$x = 0$ to $x = a$
AB	$x = a$	$y = 0$ to $y = b$
BC	$y = b$	$x = a$ to $x = 0$
CO	$x = 0$	$y = b$ to $y = 0$

Therefore, $\int_c \vec{F} \cdot d\vec{r} = \int_{OABCO} \vec{F} \cdot d\vec{r}$

$$\int_c \vec{F} \cdot d\vec{r} = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$\vec{F} \cdot d\vec{r} = (x^2 - y^2) + 2xydy \quad \ldots(2)$$

On OA: $y = 0, dy = 0, x$ varies from 0 to a

(2) $\Rightarrow \vec{F}\,.\,\mathrm{d}\vec{r} = x^2dx$

$$\int_{OA} \vec{F} \cdot d\vec{r} = \int_0^a x^2\, dx$$

$$= \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

On AB: $x = a, dx = 0, y$ varies from 0 to b

(2) $\Rightarrow \vec{F} \,.\, d\vec{r} = 2ay\,dy$

$$\int_{AB} \vec{F} \cdot d\vec{r} = \int_0^b 2ay\,dy$$

$$= \left[\frac{2ay^2}{2}\right]_0^b = ab^2$$

On BC: $y = b, dy = 0, x$ varies from a to 0

(2) $\Rightarrow \vec{F} \,.\, d\vec{r} = (x^2 - b^2)dx$

$$\int_{BC} \vec{F} \cdot d\vec{r} = \int_a^0 x^2 - b^2\,dx$$

$$= \left[\frac{x^3}{3} - b^2x\right]_a^0$$

$$= -\frac{a^3}{3} + a\,b^2$$

On CO: $x = 0, dx = 0, y$ varies from b to 0

(2) $\Rightarrow \vec{F} \,.\, d\vec{r} = 0$

$$\int_{CO} \vec{F} \cdot d\vec{r} = 0$$

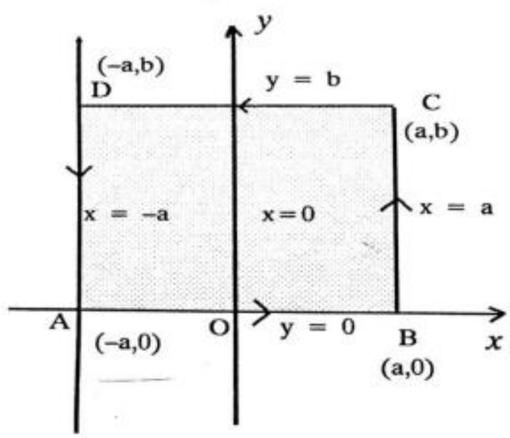
(2) $\Rightarrow \vec{F} \,.\, d\vec{r} = \frac{a^3}{3} + ab^2 - \frac{a^3}{3} + ab^2 = 2ab^2$ ... (3)

From (3) and (1) $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Hence Stokes theorem is verified.

Example: 2.73 Verify Stoke's theorem for $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$ taken around the rectangle bounded by the lines $x = \pm a, y = 0, y = b$.

Solution:



By Stokes theorem, $\int_c \vec{F} \cdot d\vec{r} = \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS$

Given $\vec{F} = (x^2 + y^2)\vec{i} - 2xy\,\vec{j}$

$$Curl\,\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= \vec{i}[0-0] - \vec{j}\,[0-0] + \vec{k}\,[-2y-2y]$$

$$= -4y\,\vec{k}$$

Since the region is in xoy plane we can take $\hat{n} = \vec{k}$ and $dS = dx\,dy$

Limits:

x varies from – a to a.

y varies from 0 to b.

$$\therefore \iint_S Curl\,\vec{F} \cdot \hat{n}\, dS = -4\int_0^b \int_{-a}^{a} y\,dx\,dy$$

$$= -4\int_0^b [xy]_{-a}^{a}\, dy$$

$$= -8a\left[\frac{y^2}{2}\right]_0^b = -4ab^2 \quad ...(1)$$

$$\int_c \vec{F}\cdot d\vec{r} = \int_{AB} + \int_{BC} + \int_{CD} + \int_{DA}$$

Along AB: $y = 0, dy = 0, x$ varies from $-a\ to\ a$

$$d\vec{r} = dx\,\vec{i} + dy\,\vec{j}$$

$$\int_{AB} \vec{F}\cdot d\vec{r} = \int_{-a}^{a} x^2\,dx$$

$$= \left[\frac{x^3}{3}\right]_{-a}^{a} = \frac{2a^3}{3}$$

Along $BC, x = a, dx = 0,$ y varies from $0\ to\ b$

$$\int_{BC} \vec{F}\cdot d\vec{r} = \int_0^b (-2ay)\,dy$$

$$= -a[y^2]_0^b = -ab^2$$

Along CD: $y = b, dy = 0, x$ varies from $a\ to -a$

$$\int_{CD} \vec{F}\cdot d\vec{r} = \int_a^{-a} (x^2 + b^2)\,dx = \left[\frac{x^3}{3} + b^2x\right]_a^{-a}$$

$$= -\frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2 = -\frac{2a^3}{3} - 2ab^2$$

Along DC: $x = -a, dx = 0, y$ varies from $b\ to\ 0$

$$\int_{DC} \vec{F}\cdot d\vec{r} = \int_b^0 2ay\,dy$$

$$= a[y^2]_b^0 = -b^2a$$

$$\therefore \int_c \vec{F}\cdot d\vec{r} = \frac{2a^3}{3} - ab^2 - \frac{2a^3}{3} - 2ab^2 - b^2a$$

$$= -4ab^2 \quad \ldots.(2)$$

From (1) and (2) $\int_c \vec{F}\cdot d\vec{r} = \iint_S Curl\,\vec{F}\cdot\vec{n}\,dS$

Hence Stoke's theorem is verified.