Change of order of integration

Change of order of integration is done to make the evaluation of integral easier

The following are very important when the change of order of integration takes place

1. If the limits of the inner integral is a function of x (or function of y) then the first integration should be with respect to y (or with respect to x)
2. Draw the region of integration by using the given limits
3. If the integration is first with respect to x keeping y as a constant then consider the horizontal strip and find the new limits accordingly
4. If the integration is first with respect to y keeping x a constant then consider the vertical strip and find the new limits accordingly
5. After find the new limits evaluate the inner integral first and then the outer integral

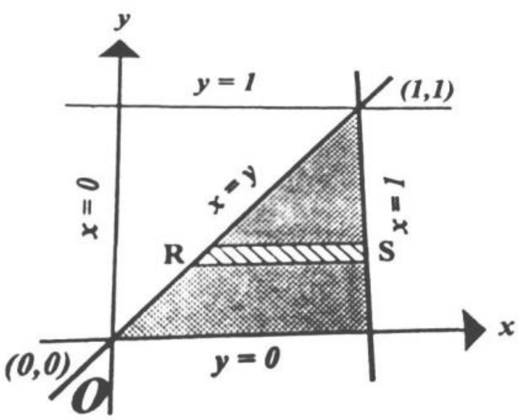
Sketch roughly the region of integration for $\int_0^1 \int_0^x f(x,y)\, dydx$

Solution:

Given $\int_0^1 \int_0^x f(x,y)\, dydx$

x varies from $x = 0$ to $x = 1$

y varies from $y = 0$ to $y = x$



Example: 4.12

Shade the region of integration $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dxdy$

Solution:

$$\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} dydx \text{ is the correct form}$$

x limit varies from $x = 0$ to $x = a$

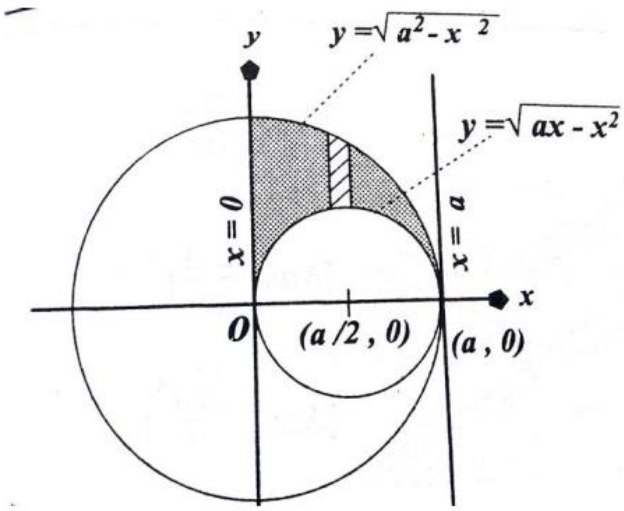
y limit varies from $y = \sqrt{ax - x^2}$ to $y = \sqrt{a^2 - x^2}$

i.e., $y^2 = ax - x^2$ to $y^2 = a^2 - x^2$

i.e., $y^2 + x^2 = ax$ to $y^2 + x^2 = a^2$

$x^2 + y^2 = ax$ is a circle with centre $\left(\frac{a}{2}, 0\right)$ and radius $\frac{a}{2}$

$x^2 + y^2 = a^2$ is a circle with centre $(0,0)$ and radius a



Example:4.18

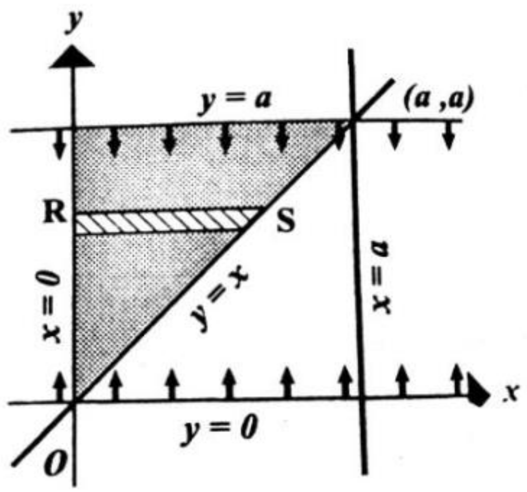
Change the order of integration in $\int_0^a \int_x^a f(x, y) dy\, dx$

Solution:

Given $y: x \to a$

$x: 0 \to a$

The region is bounded by $y = x, y = a, x = 0$ and $x = a$



x axis limit represents the horizontal strip and y axis limit represents vertical

$x: 0 \to y$

$y: 0 \to a$

By changing the order we get

$$\int_0^a \int_0^y f(x,y)dx\,dy$$

Example: 4.19

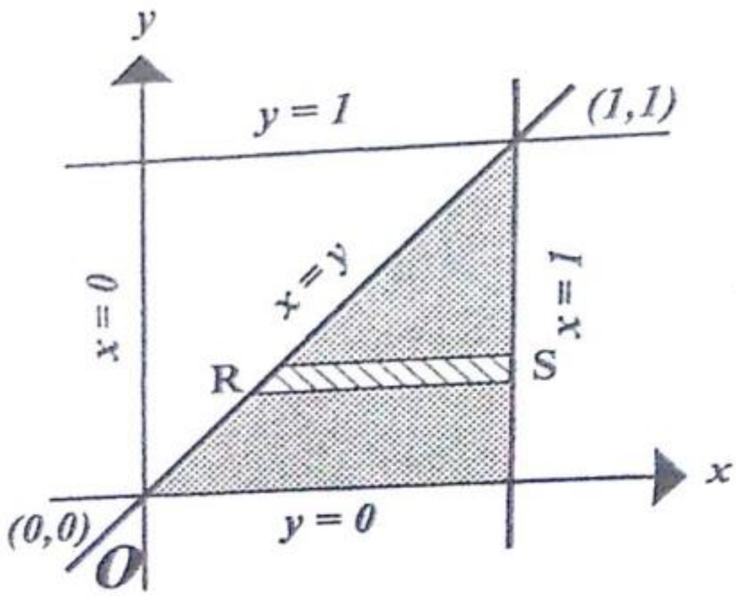
Change the order of integration $\int_0^1 \int_0^x f(x,y)dy\,dx$

Solution:

Given $y: 0 \to x$

$x: 0 \to 1$

The region is bounded by $y = 0, y = x, x = 0, x = 1$



$x: y \to 1$

$y: 0 \to 1$

By changing the order we get

$$\int_0^1 \int_y^1 f(x,y)dx\,dy$$

Example: 4.20

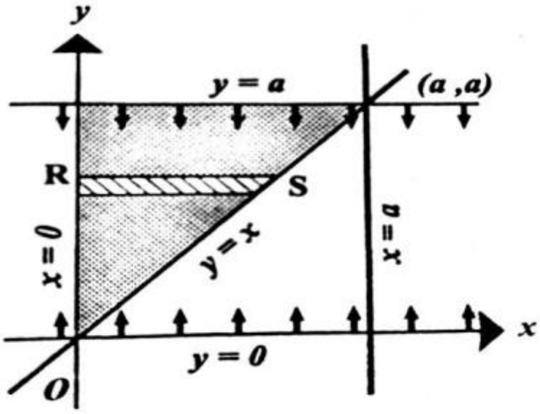
Change the order of integration and hence evaluate it $\int_0^a \int_x^a (x^2+y^2)dy\,dx$

Solution:

It is correct form, given order is $dydx$ given $y: x \to a$

$x: 0 \to a$

the region is bounded by $y = x, y = a\ x = 0$ and $x = a$



x axis limit represent the horizontal strip

y axis limit represents vertical path

changed order is $dx\,dy$

$x: 0 \to y$

$y: 0 \to a$

$$\int_0^a \int_0^y (x^2+y^2)dx\,dy = \int_0^a \left[\frac{x^3}{3} + y^2 x\right]_0^y dy$$

$$= \int_0^a \left[\frac{y^3}{3} + y^3\right] dy$$

$$= \left[\frac{y^4}{12} + \frac{y^4}{4}\right]_0^a \quad = \frac{a^4}{12} + \frac{a^4}{4} = \frac{a^4}{3}$$

Example: 4.21

Change the order of integration for $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} xy\,dy\,dx$

Solution:

It is correct form

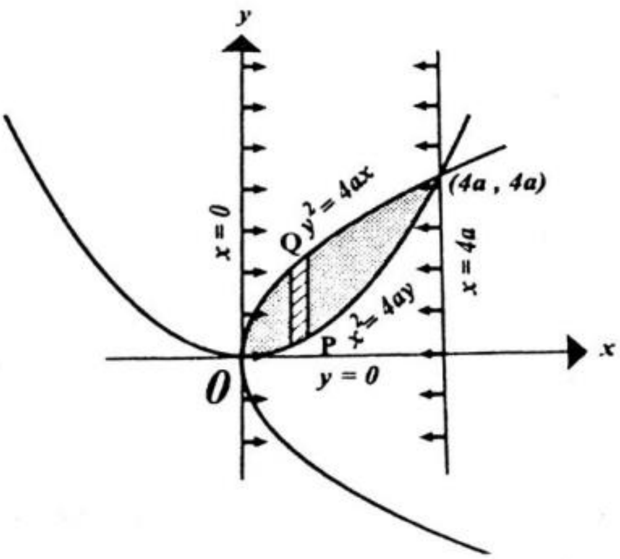
Given order is $dydx$

Given $y: \frac{x^2}{4a} \to 2\sqrt{ax}$

$x: 0 \to 4a$

The region is bounded by $x^2 = 4ay$, $y^2 = 4ax$

$x = 0$ and $x = 4a$



Changed order is $dxdy$ draw a horizontal strip

$x: \frac{y^2}{4a} \to 2\sqrt{ay}$

$y : 0 \to 4a$

$$\int_0^{4a}\int_{y^2/4a}^{2\sqrt{ay}} xy\,dx\,dy = \int_0^{4a} \left[\frac{x^2y}{2}\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a} \left\{\frac{(2\sqrt{ay})^2 y}{2} - [\frac{y^2}{4a}]^2 \frac{y}{2}\right\} \; dy$$

$$= \int_0^{4a} \left[\left(\frac{4ay}{2}\right) y - \frac{y^5}{32a^2}\right] \; dy$$

$$= \left[\frac{4ay^3}{6} - \frac{y^6}{192a^2}\right]_0^{4a}$$

$$= \frac{4a(4a)^3}{6} - \frac{(4a)^6}{192a^2}$$

$$= \frac{128a^4}{3} - \frac{4096}{192} a^4$$

$$= \frac{64a^4}{3}$$

Example: 4.22

Change the order of integration of $\int_0^b \int_0^{\frac{a}{b}\sqrt{b^2-y^2}} xydxdy$ and hence evaluate it

Solution:

It is correct form

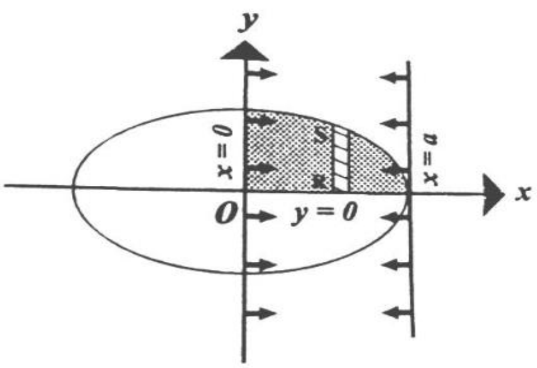
Given order is $dxdy$

Given $x : 0 \to \frac{a}{b}\sqrt{b^2 - y^2}$

$y : \; 0 \to b$

The region is bounded by $x = 0, x = \frac{a}{b}\sqrt{b^2 - y^2} \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$y = 0, y = b$



Changed order is $dydx$

Draw the vertical strip

$y : 0 \to \frac{b}{a}\sqrt{a^2 - x^2}$

$x : 0 \to a$

$$\int_0^a \int_0^{\frac{b}{a}\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \left[\frac{xy^2}{2}\right]_0^{\frac{b}{a}\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \frac{\left[\frac{b}{a}\sqrt{a^2-x^2}\right]^2 x}{2}\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a x(a^2 - x^2)\, dx$$

$$= \frac{b^2}{2a^2}\int_0^a (xa^2 - x^3)\text{dx}$$

$$= \frac{b^2}{2a^2}\left[\frac{a^2x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$= \frac{b^2}{2a^2}\left[\frac{a^4}{2} - \frac{a^4}{4}\right]$$

$$= b^2\left[\frac{a^4}{2a^2} - \frac{a^4}{4}\right] = b^2\left[\frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^2b^2}{8}$$

Example: 4.23

Change the order of integration and hence evaluate $\int_0^1 \int_{x^2}^{2-x} xy\, dy\, dx$

Solution:

It is correct form

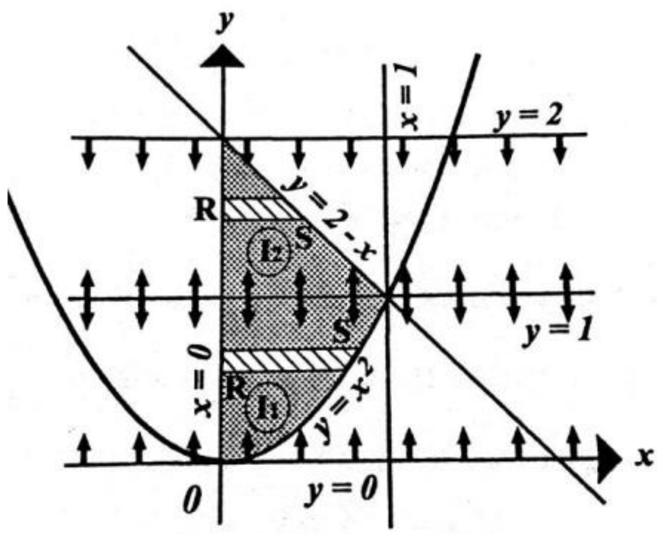
Given order is $dydx$

Given $y: x^2 \rightarrow 2 - x$

$x: 0 \rightarrow 1$

The region is bounded by $= x^2, y + x = 2$

$x = 0, x = 1$



Now divide the region in to two parts i.e. R_1 and R_2

Changed order is $dxdy$

Draw horizontal strip

For Region R_1

Limits are $x: 0 \rightarrow \sqrt{y}$

$y: 0 \rightarrow 1$

$$\int_0^1 \int_0^{\sqrt{y}} xy\, dx\, dy = \int_0^1 \left[\frac{x^2 y}{2}\right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \frac{(\sqrt{y})^2 y}{2} dy$$

$$= \int_0^1 \frac{y^2}{2} dy$$

$$= \left[\frac{y^3}{6}\right]_0^1$$

$$= 1/6$$

For region R_2

Limits are $x: 0 \to 2-y$

$y: 1 \to 2$

$$\int_1^2 \int_0^{2-y} xy\,dx\,dy = \int_1^2 \left[\frac{x^2 y}{2}\right]_0^{2-y} dy$$

$$= \int_1^2 \frac{(2-y)^2 y}{2}\,dy$$

$$= \int_1^2 \frac{(4-4y+y^2)y}{2}\,dy$$

$$= \frac{1}{2}\left[\frac{4y^2}{2} - \frac{4y^3}{3} + \frac{y^4}{4}\right]_1^2$$

$$= \frac{1}{2}\left[8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right]$$

$$= \frac{5}{24}$$

$$R = R_1 + R_2$$

$$= \frac{1}{6} + \frac{5}{24}$$

$$= \frac{3}{8}$$

Example: 4.24

Change the order of integration in $\int_0^1 \int_y^{2-y} xy\,dx\,dy$ and hence evaluates

Solution:

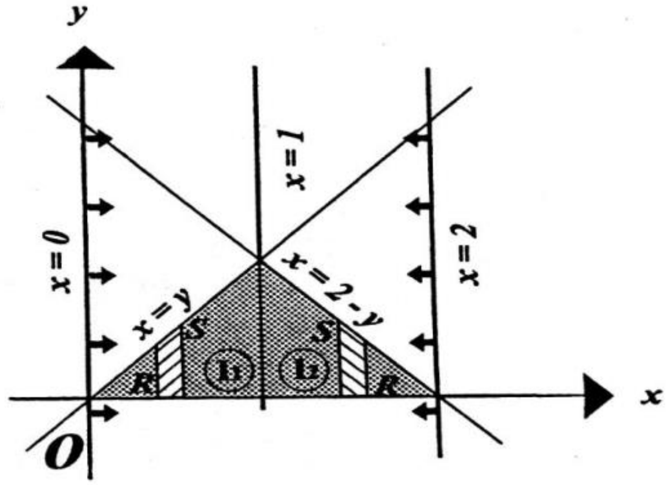
It is correct form

$x: y \to 2-y$

$y: 0 \to 1$

The region is bounded by $x = y$, $x + y = 2$

$y = 0$, $y = 1$



Now divide the region into two parts ie. R_1 and R_2

Changed order is $dydx$

Draw horizontal strip

For region R_1

Limits are $x: 0 \to 1$

$y: 0 \to x$

$$\int_0^1 \int_y^{2-y} xy\, dx\, dy = \int_0^1 \int_0^x xy\, dy\, dx$$

$$= \int_0^1 \left[\frac{xy^2}{2}\right]_0^x dx$$

$$= \int_0^1 \left[\frac{x^3}{3-0}\right] dx$$

$$= \frac{1}{2}\int_0^1 x^3\, dx = \frac{1}{2}\left[\frac{x^4}{4}\right]_0^1$$

$$= \frac{1}{8}[x^4]_0^1 = \frac{1}{8}[1-0]$$

$$= \frac{1}{8}$$

For region R_2

$x: 1 \to 2$

$y: 0 \to 2 - x$

$$\int_1^2 \int_0^{2-x} xy\, dy\, dx = \int_1^2 \left[\frac{xy^2}{2}\right]_0^{2-x} dx$$

$$= \int_1^2 \left[\frac{x\,(2-x)^2}{2} - 0\right] dx$$

$$= \frac{1}{2}\int_1^2 \frac{x(4+x^2-4x)}{2}\, dx$$

$$= \frac{1}{2}\int_1^2 (4x + x^3 - 4x^2)\, dx$$

$$= \frac{1}{2}\left[4\frac{x^2}{2} + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[2x^2 + \frac{x^4}{4} - 4\frac{x^3}{3}\right]_1^2$$

$$= \frac{1}{2}\left[\left(8 + \frac{16}{4} - \frac{4}{3}(8)\right) - \left(2 + \frac{1}{4} - \frac{4}{3}\right)\right]$$

$$= \frac{1}{2}\left[8 + 4 - \frac{32}{3} - 2 - \frac{1}{4} + \frac{4}{3}\right]$$

$$= \frac{1}{2}\left[\frac{5}{12}\right] = \frac{5}{24}$$

$$\Rightarrow R = R_1 + R_2$$

$$= \frac{1}{8} + \frac{5}{24}$$

$$= \frac{1}{3}$$

Example: 4.25

Change the order of integration $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ and hence evaluate it

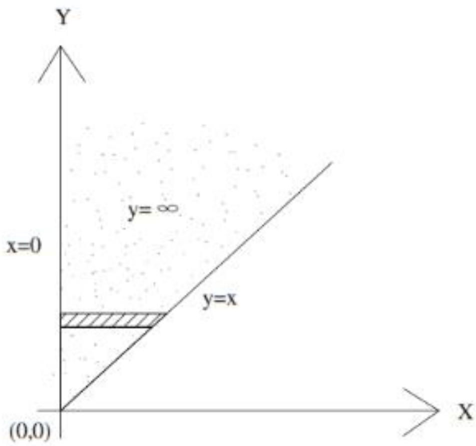
Solution:

It is correct form

Given order is $dy\,dx$

Given $y: \; x \to \infty$

$x: \; 0 \to \infty$



Changed order is $dxdy$

Draw a horizontal strip

$x: \; 0 \to y$

$y: \; 0 \to \infty$

$$\int_0^\infty \int_0^y \frac{e^{-y}}{y} dx\, dy = \int_o^\infty \left[e^{-y} \frac{x}{y}\right]_0^y dy$$

$$= \int_o^\infty e^{-y}\, dy = \left[\frac{e^{-y}}{1}\right]_0^\infty$$

$$= -[e^{-\infty} - e^0] = 1$$

Example: 4.26

Change the order of integration I = $\int_0^a \int_y^a \frac{x}{x^2+y^2}\, dx\, dy$ and the evaluate it

Solution:

It is correct form

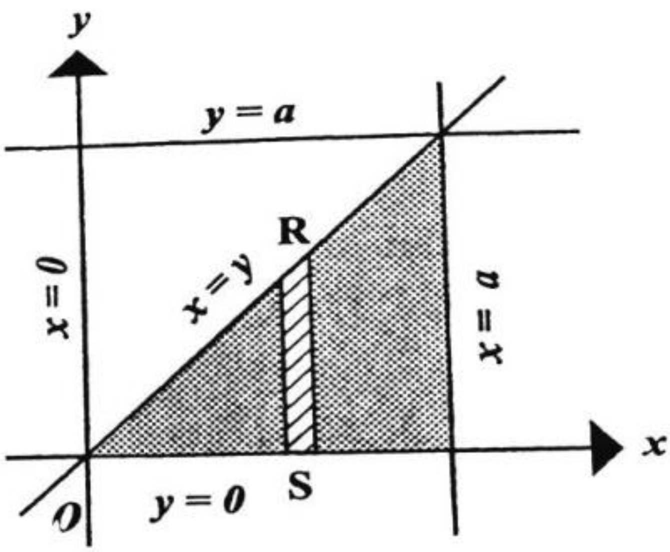
Given order $dxdy$

$x : y \to a$

$y : 0 \to a$

The region is bounded by $x = y\, , x = a$

$y = 0\, , y = a$



Changed order is $dy\, dx$

Draw a vertical strip

$y : 0 \to x$

$x : 0 \to a$

$$\int_0^a \int_0^x \frac{x}{x^2+y^2}\, \text{dy dx} = \int_0^a x \left[\frac{\tan^{-1}}{x}\left(\frac{y}{x}\right)\right]_0^x dx$$

$$= \int_0^a [tan^{-1}\left(\frac{x}{x}\right) - tan^{-1}0]\, dx$$

$$= \int_0^a \frac{\pi}{4}\, dx$$

$$= \left[\frac{\pi}{4} x\right]_0^a$$

$$= \frac{\pi}{4} a$$

Example: 4.27

Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx$ by changing the order of integration

Solution:

It is correct form

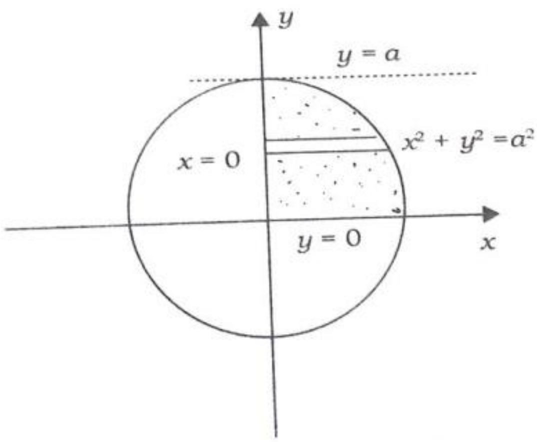
Given order $dy\, dx$

Given $y:\ 0 \to \sqrt{a^2 - x^2}$

$x:\ 0 \to a$

the region is bounded by $y = 0,\ y = \sqrt{a^2 - x^2}$

$x = 0\ ,\ x = a$



changed order $dxdy$

Draw horizontal strip

$$x : 0 \to \sqrt{a^2 - y^2}$$

$$y : 0 \to a$$

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} xy\, dy\, dx = \int_0^a \int_0^{\sqrt{a^2-y^2}} xy\, dx\, dy$$

$$= \int_0^a y \left[\frac{x^2}{2}\right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \frac{1}{2} \int_0^a y(a^2 - y^2)\, dy$$

$$= \frac{1}{2} \left[\frac{a^2 y^2}{2} - \frac{y^4}{4}\right]_0^a$$

$$= \frac{1}{2}\left[\frac{a^2}{2} - \frac{a^4}{4}\right]$$

$$= \frac{a^4}{8}$$