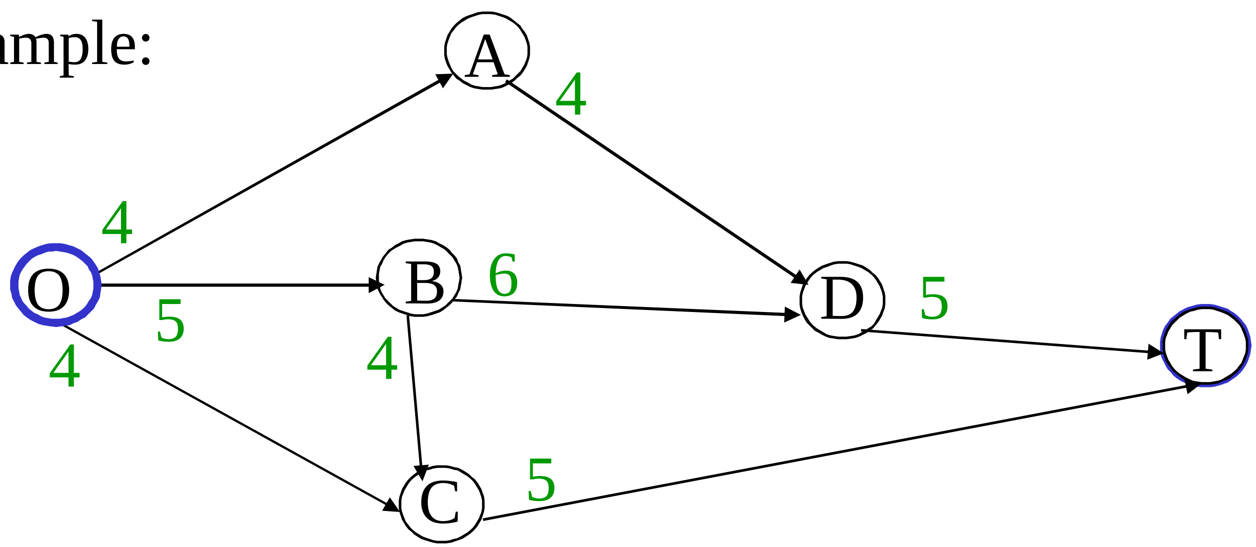
Network Optimization Models: Maximum Flow Problems

In this handout:

- The problem statement
- Augmenting path algorithm for solving the problem

Maximum Flow Problem

- *Given*: Directed graph G=(V, E),
  Supply (source) node O, demand (sink) node T
  Capacity function $u: E \to R$.
- *Goal*: Given the arc capacities,
  send as much flow as possible
  from supply node O to demand node T
  through the network.
- Example:



Towards the Augmenting Path Algorithm

- *Idea*: Find a path from the source to the sink,

  and use it to send as much flow as possible.

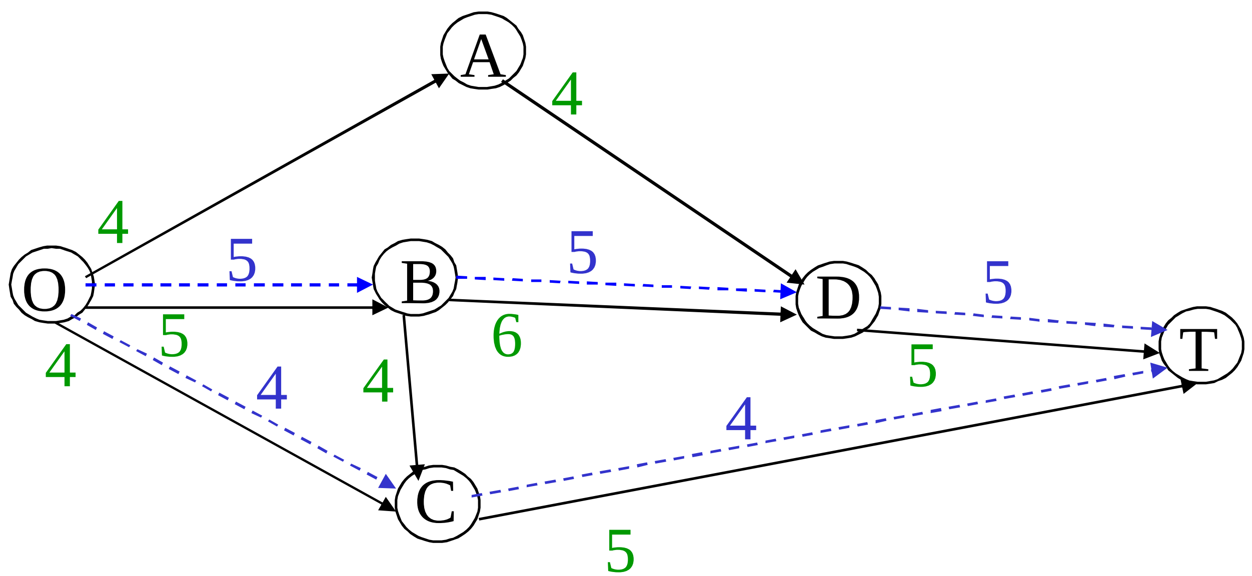
- In our example,

  5 units of flow can be sent through the path O → B → D → T ;

  Then use the path O → C → T to send 4 units of flow.

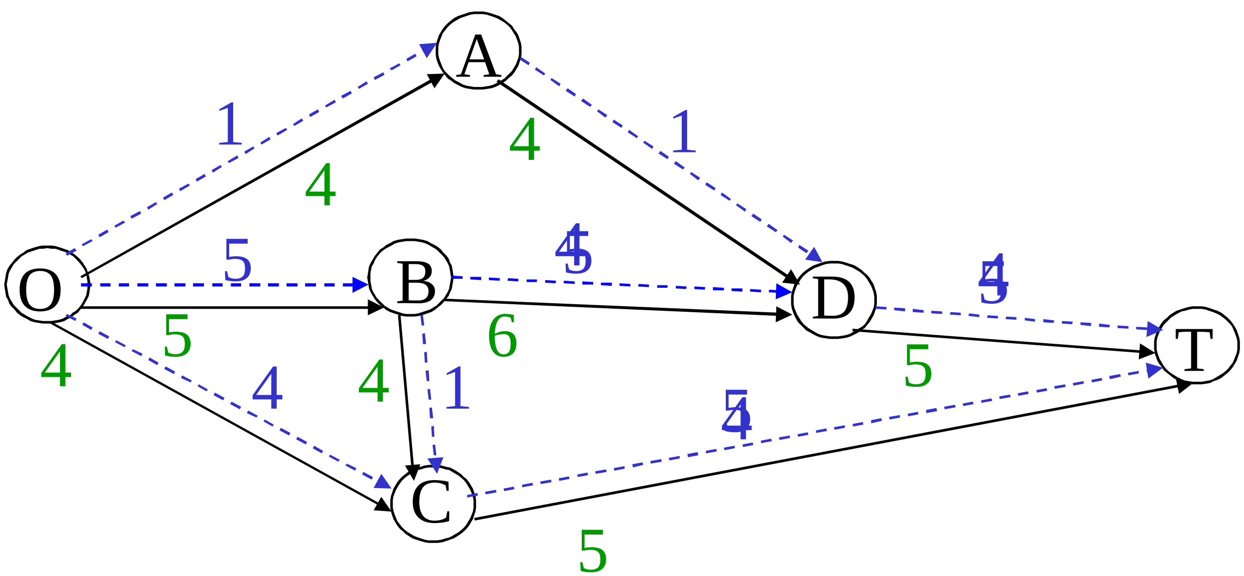
  The total flow is 5 + 4 = 9 at this point.

- Can we send more?



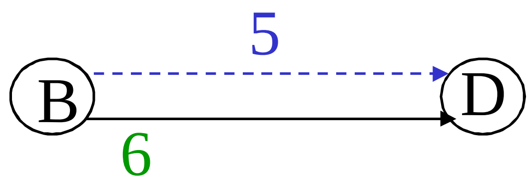
Towards the Augmenting Path Algorithm

- If we redirect 1 unit of flow
  from path O → B → D → T to path O → B → C → T,
  then the freed capacity of arc D → T could be used
  to send 1 more unit of flow through path O → A → D → T,
  making the total flow equal to 9+1=10 .
- To realize the idea of redirecting the flow in a systematic way,
  we need the concept of *residual capacities*.

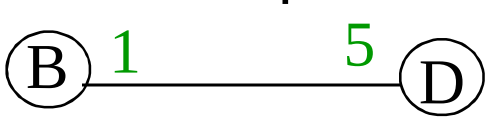


Residual capacities

- Suppose we have an arc with capacity 6 and current flow 5:



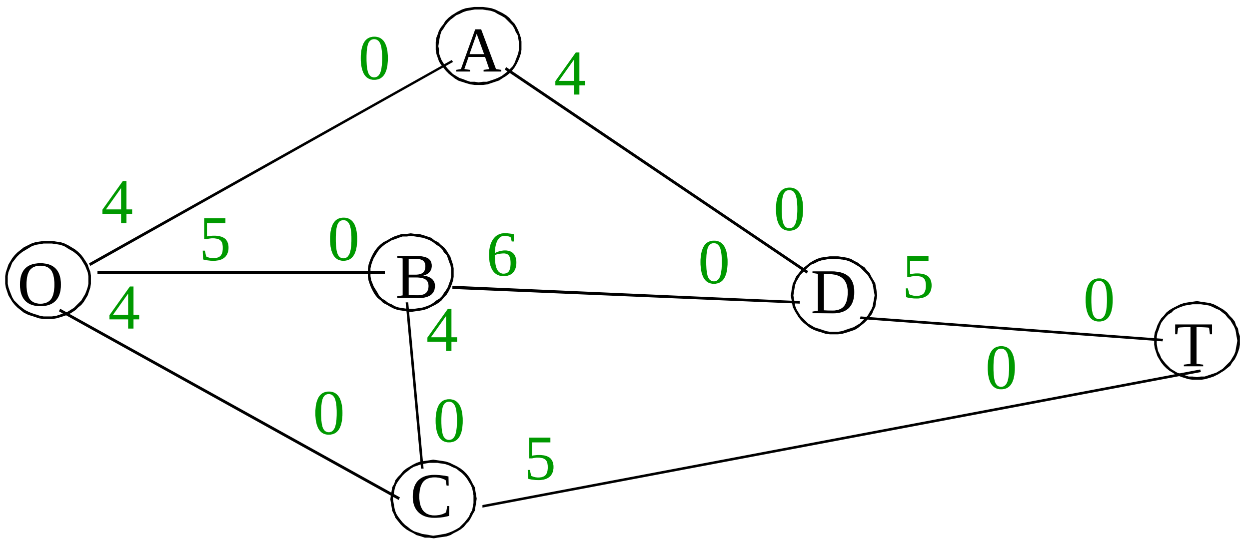
- Then there is a *residual capacity* of 6-5=1
  for any additional flow through B → D .
- On the other hand,
  at most 5 units of flow can be sent back from D to B, i.e.,
  5 units of previously assigned flow can be canceled.
  In that sense, 5 can be considered as
  the residual capacity of the reverse arc D → B .
- To record the residual capacities in the network,
  we will replace the original directed arcs with undirected arcs:



The number at B is the residual capacity of B→D;
the number at D is the residual capacity of D→B.

Residual Network

- The network given by the undirected arcs and residual capacities is called *residual network.*
- In our example, the residual network before sending any flow:



Note that the sum of the residual capacities on both ends of an arc is equal to the original capacity of the arc.

- How to increase the flow in the network based on the values of residual capacities?

Augmenting paths

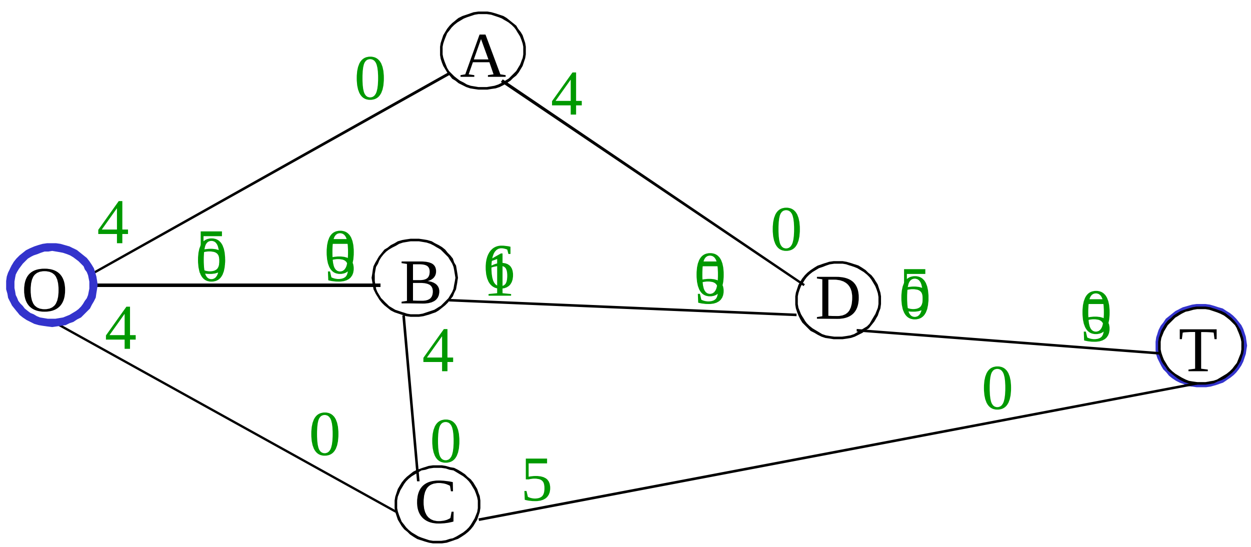
- An *augmenting path* is a directed path
  from the source to the sink in the residual network
  such that
  every arc on this path has positive residual capacity.
- The minimum of these residual capacities
  is called the *residual capacity of the augmenting path*.
  This is the amount
  that can be feasibly added to the entire path.
- The flow in the network can be increased
  by finding an augmenting path
  and sending flow through it.

Updating the residual network by sending flow through augmenting paths

Continuing with the example,

- *Iteration 1*: O → B → D → T is an augmenting path with residual capacity 5 = min{5, 6, 5}.
- After sending 5 units of flow through the path O → B → D → T,

the new residual network is:



Updating the residual network by sending flow through augmenting paths

- *Iteration 2*:

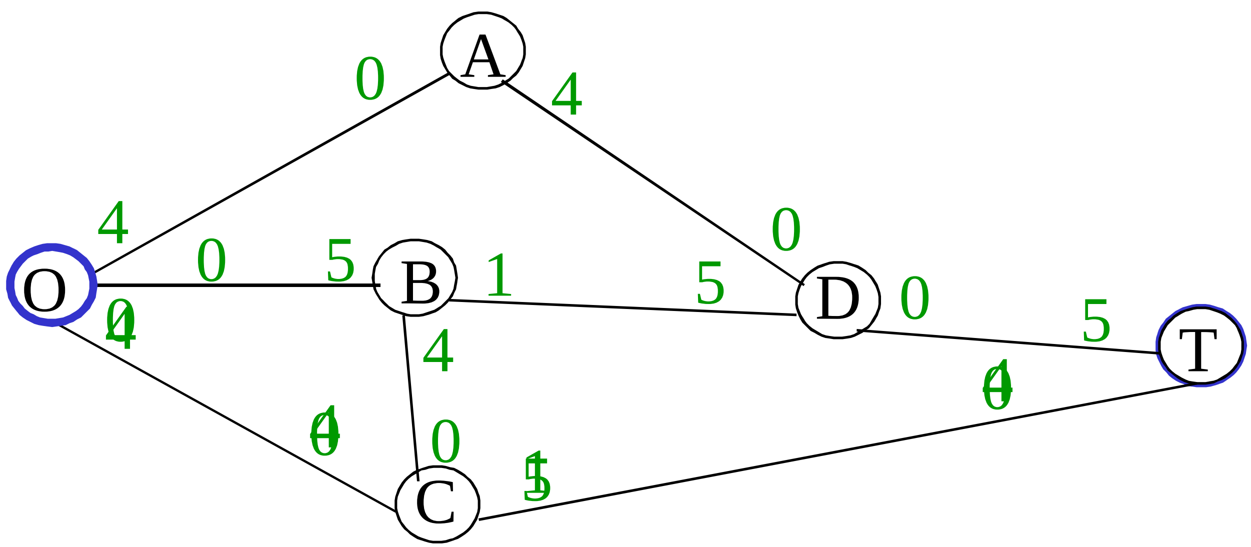
  O → C → T is an augmenting path

  with residual capacity 4 = min{4, 5}.

- After sending 4 units of flow

  through the path O → C → T,

  the new residual network is:



Updating the residual network by sending flow through augmenting paths

- *Iteration 3*:

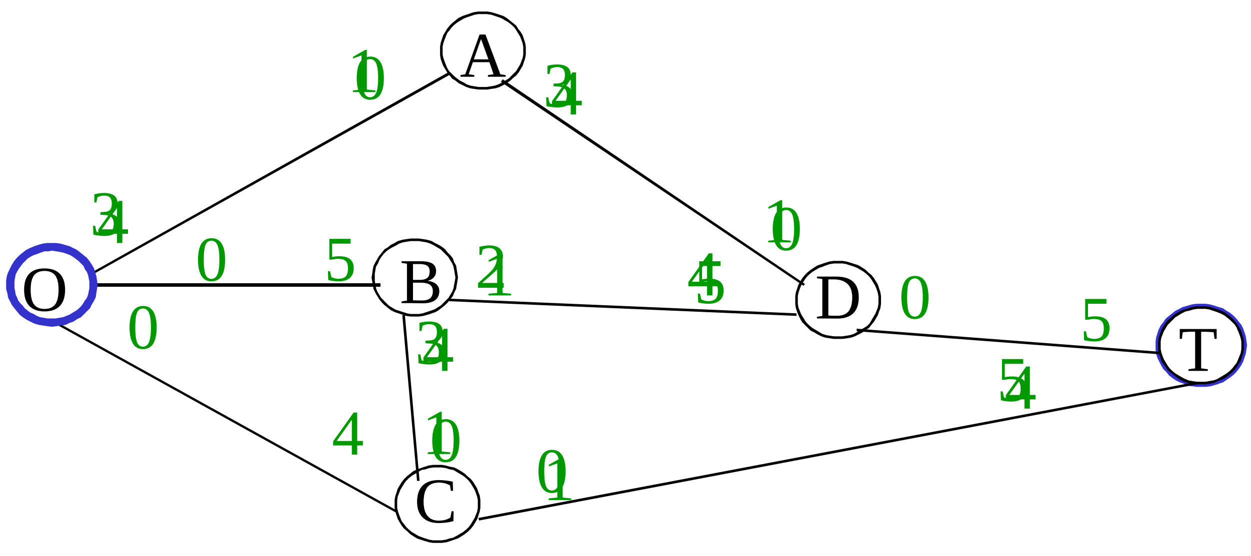
  O → A → D → B → C → T is an augmenting path

  with residual capacity 1 = min{4, 4, 5, 4, 1}.

- After sending 1 units of flow

  through the path O → A → D → B → C → T ,

  the new residual network is:



Terminating the Algorithm: Returning an Optimal Flow

- There are no augmenting paths in the last residual network.

  So the flow from the source to the sink cannot be increased further, and the current flow is optimal.
- Thus, the current residual network is optimal.

  The optimal flow on each directed arc of the original network is the residual capacity of its reverse arc:

  flow(O→A)=1, flow(O→B)=5, flow(O→C)=4,

  flow(A→D)=1, flow(B→D)=4, flow(B→C)=1,

  flow(D→T)=5, flow(C→T)=5.

  The amount of maximum flow through the network is

  $5 + 4 + 1 = 10$

  (the sum of path flows of all iterations).

The Summary of the Augmenting Path Algorithm

- **Initialization**: Set up the initial residual network.
- **Repeat**
  - Find an augmenting path.
  - Identify the residual capacity c* of the path; increase the flow in this path by c*.
  - Update the residual network: decrease by c* the residual capacity of each arc on the augmenting path; increase by c* the residual capacity of each arc in the opposite direction on the augmenting path.

  Until no augmenting path is left
- **Return** the flow corresponding to

  the current optimal residual network