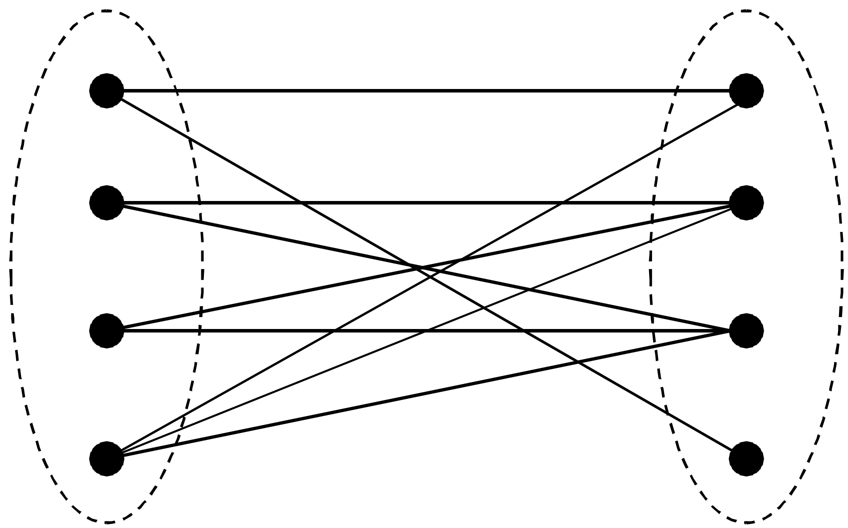
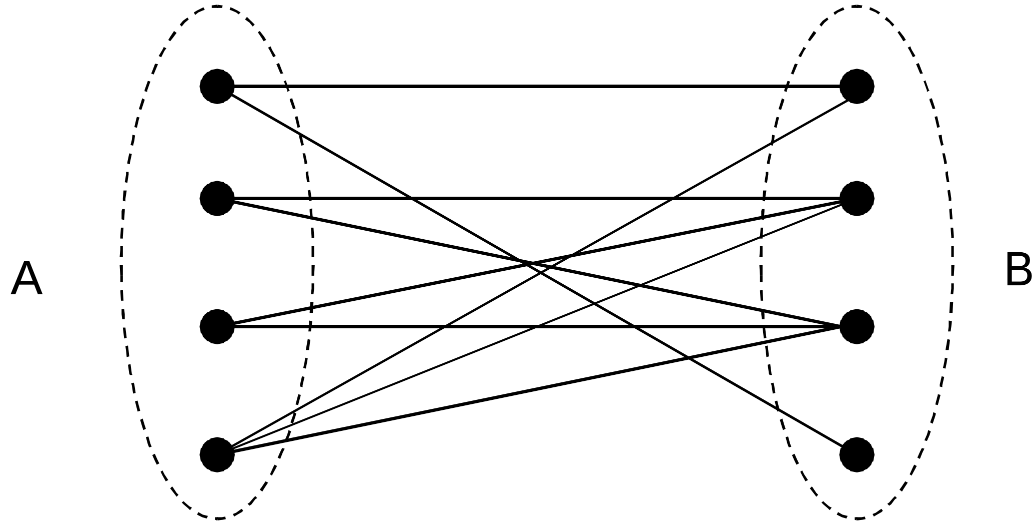
Bipartite Matching



Bipartite Matching

A graph is **bipartite** if its vertex set can be partitioned into two subsets A and B so that each edge has one endpoint in A and the other endpoint in B.

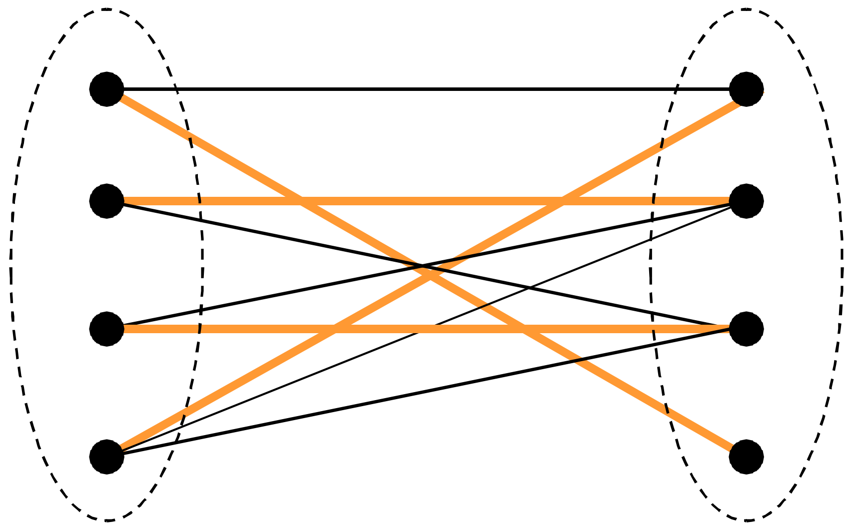


A **matching** M is a subset of edges so that every vertex has degree at most one in M.

Maximum Matching

The bipartite matching problem:

Find a matching with the maximum number of edges.



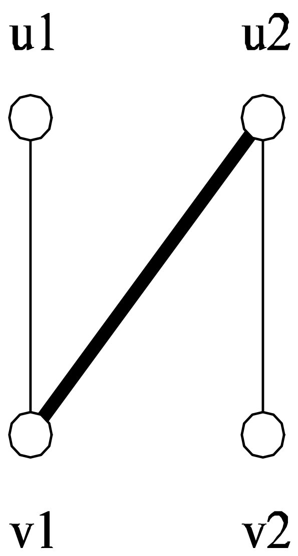
A perfect matching is a matching in which every vertex is matched.

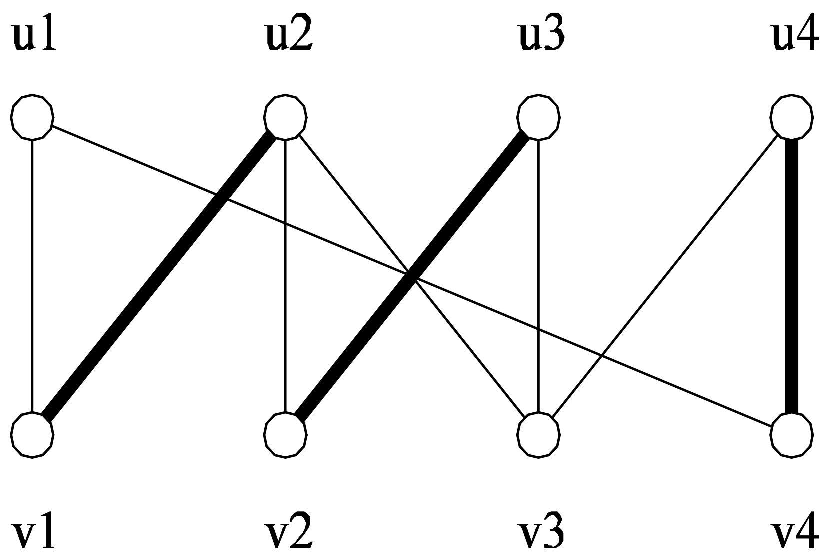
The perfect matching problem: Is there a perfect matching?

First Try

- Greedy method?

  (add an edge with both endpoints *unmatched*)





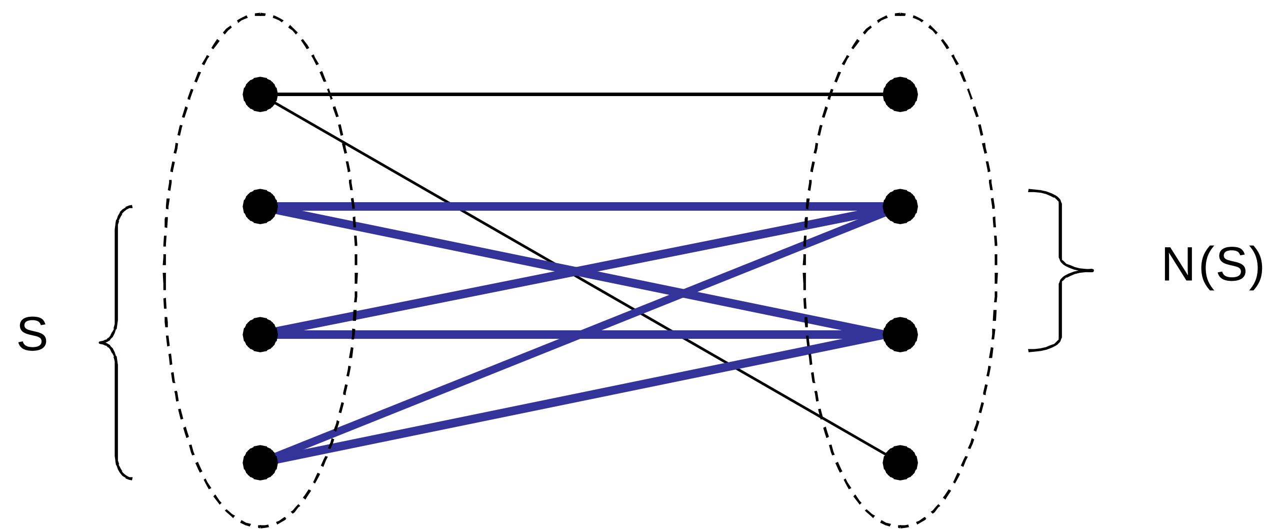
Key Questions

- How to tell if a graph does not have a (perfect) matching?
- How to determine the size of a maximum matching?
- How to find a maximum matching efficiently?

Existence of Perfect Matching

Hall's Theorem [1935]:

A bipartite graph G=(A,B;E) has a matching that "saturates" A if and only if |N(S)| >= |S| for every subset S of A.



Bound for Maximum Matching

What is a good upper bound on the size of a maximum matching?

König [1931]:

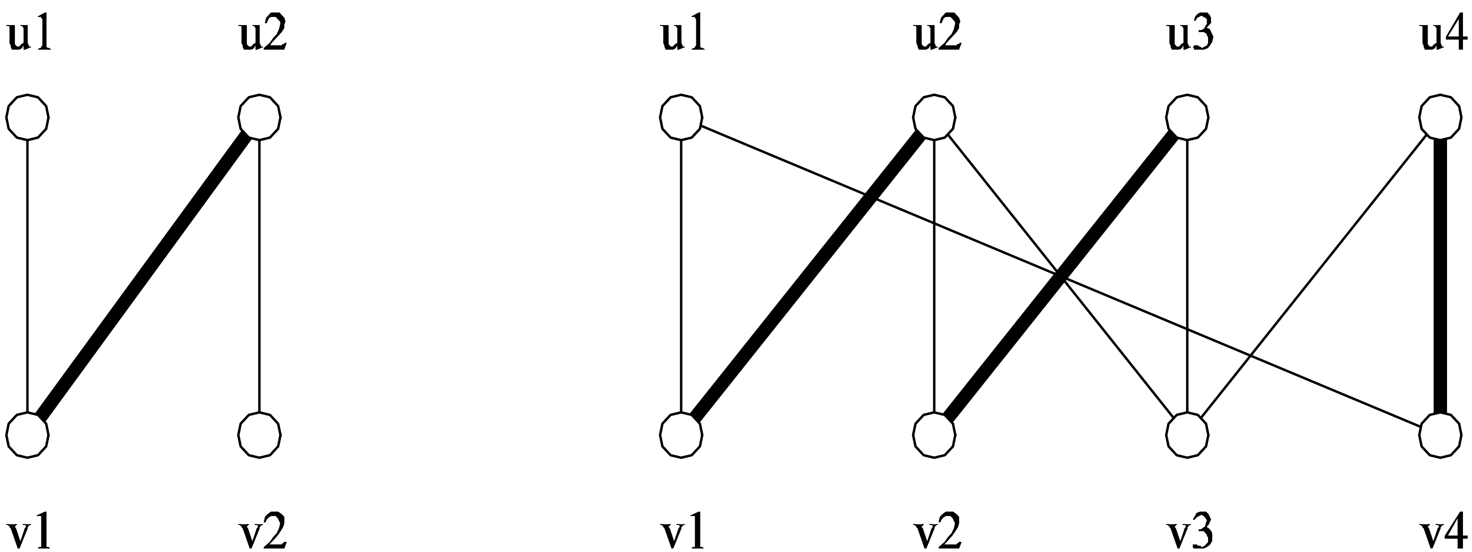
In a bipartite graph, the size of a maximum matching is **equal** to the size of a minimum vertex cover.

Min-max theorem

NP and co-NP

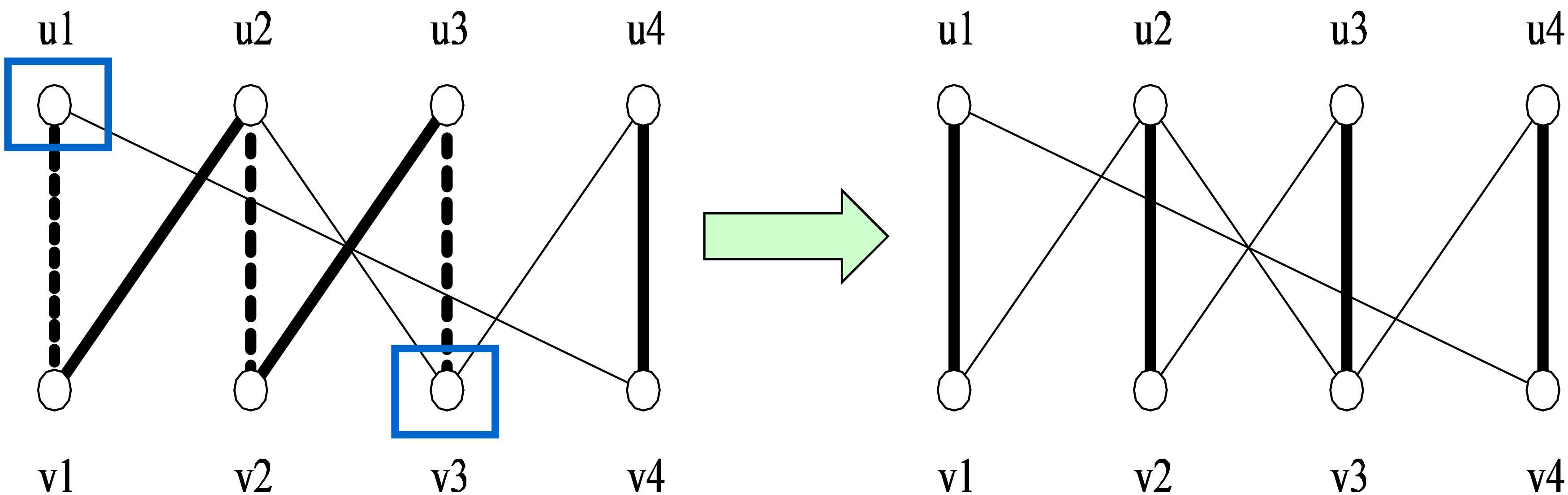
Implies Hall's theorem.

Algorithmic Idea?



Any idea to find a larger matching?

Augmenting Path



Given a matching M, an **M-alternating path** is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are unmatched by M is an **M-augmenting path**.

$$M^* = M \oplus P$$

Optimality Condition

What if there is no more M-augmenting path?

If there is no M-augmenting path, then M is maximum!

Prove the contrapositive:

A bigger matching ⇨ an *M*-augmenting path

1. Consider $H := M \cup M^*$
2. Every vertex in H has degree at most 2
3. A component in H is an *even* cycle or a path
4. Since $|M^*| > |M|$, $\exists$ an M-augmenting path!

Algorithm

Key: *M* is maximum ⇔ no *M*-augmenting path

$M := \emptyset$.

while there is an M-augmenting path P

set $M := M \oplus P$

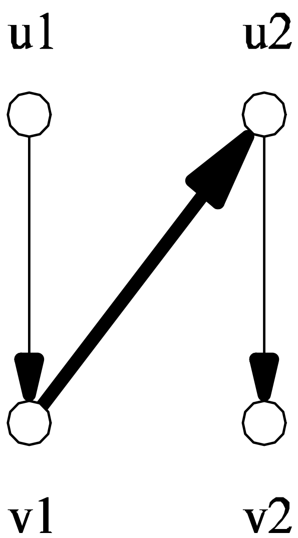
return M.

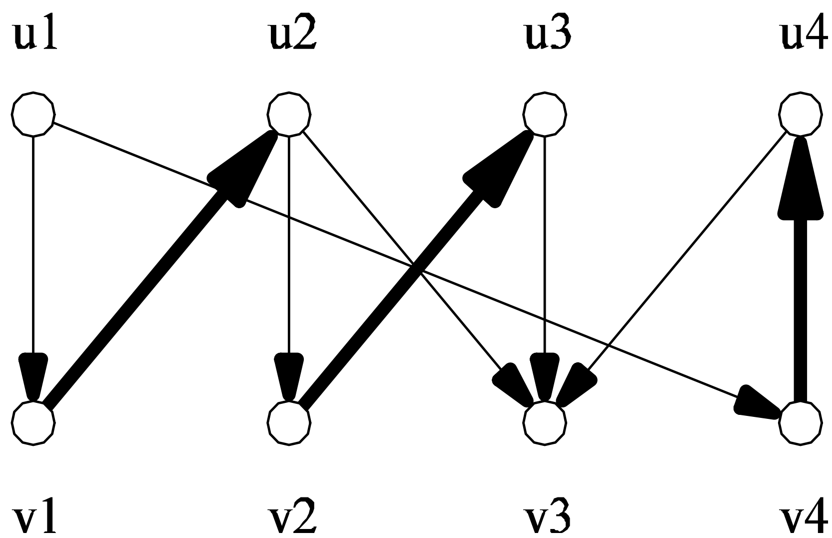
How to find efficiently?

Finding *M*-augmenting paths

- *Orient* the edges (edges in *M* go up, others go down)
- An *M*-augmenting path ⇔

  a directed path between two *unmatched* vertices





Complexity

- At most n iterations
- An augmenting path in $O(m)$ time by a DFS or a BFS
- Total running time $O(mn)$

Minimum Vertex Cover

Hall’s Theorem [1935]:

A bipartite graph G=(A,B;E) has a matching that “saturates” A

if and only if |N(S)| >= |S| for every subset S of A.

König [1931]:

In a bipartite graph, the size of a maximum matching

is **equal** to the size of a minimum vertex cover.

Idea: consider why the algorithm got stuck…

Faster Algorithms

	$O(nm)$	Kőnig [1931], Kuhn [1955b]
	$O(\sqrt{n}\,m)$	Hopcroft and Karp [1971,1973], Karzanov [1973a]
*	$\tilde{O}(n^{\omega})$	Ibarra and Moran [1981]
	$O(n^{3/2}\sqrt{\frac{m}{\log n}})$	Alt, Blum, Mehlhorn, and Paul [1991]
*	$O(\sqrt{n}\,m\log_n(n^2/m))$	Feder and Motwani [1991,1995]

Observation: Many short and disjoint augmenting paths.

Idea: Find augmenting paths simultaneously in one search.

Randomized Algorithm

- Matching
- Determinants
- Randomized algorithms

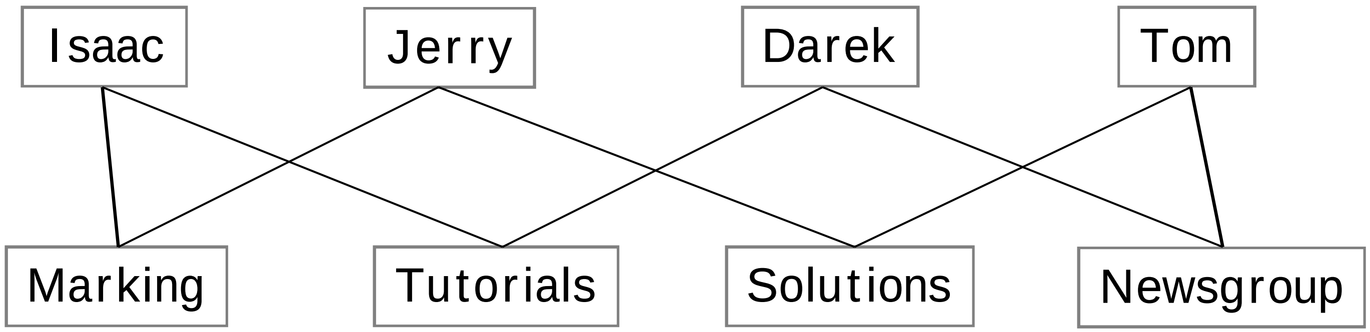
Bonus problem 1 (50%):

Given a bipartite graph with red and blue edges,

find a deterministic polynomial time algorithm to determine

if there is a perfect matching with exactly k red edges.

Application of Bipartite Matching

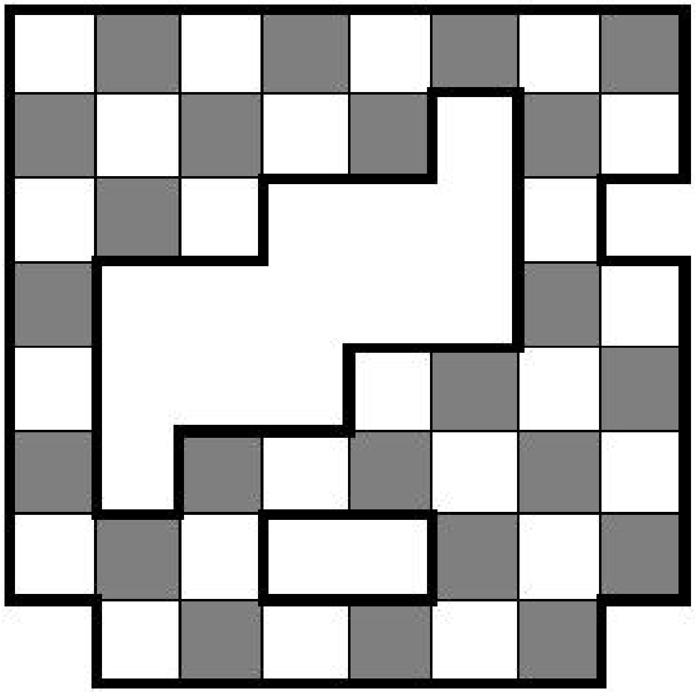


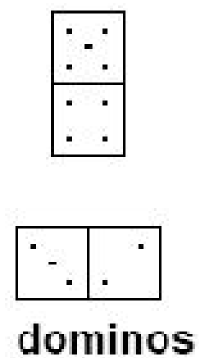
Job Assignment Problem:

Each person is willing to do a subset of jobs.

Can you find an assignment so that all jobs are taken care of?

Application of Bipartite Matching





With Hall's theorem, now you can determine exactly when a partial chessboard can be filled with dominos.

Application of Bipartite Matching

Latin Square: a nxn square, the goal is to fill the square with numbers from 1 to n so that:

- Each row contains every number from 1 to n.
- Each column contains every number from 1 to n.

1	2	3	4
3	4	2	1
2	1	4	3
4	3	1	2

Application of Bipartite Matching

Now suppose you are given a partial Latin Square.

2	4	5	3	1
4	1	3	2	5
3	2	1	5	4

Can you always extend it to a Latin Square?

With Hall's theorem, you can prove that the answer is yes.