Chapter 7

The Simplex Method

In this chapter, you will learn how to solve linear programs. This will give you insights into what SOLVER and other commercial linear programming software packages actually do. Such an understanding can be useful in several ways. For example, you will be able to identify when a problem has alternate optimal solutions (SOLVER never tells you this: it always give you only one optimal solution). You will also learn about degeneracy in linear programming and how this could lead to a very large number of iterations when trying to solve the problem.

7.1 Linear Programs in Standard Form

Before we start discussing the simplex method, we point out that every linear program can be converted into "standard" form

$$
\begin{array}{rrcl}
 & \text{Max } c_1x_1 + c_2x_2 + \ldots + c_nx_n & & \\
\text{subject to} & a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & = & b_1 \\
 & \ldots \quad \ldots & & \ldots \\
 & a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n & = & b_m \\
 & x_1 \geq 0, \ldots x_n \geq 0 & &
\end{array}
$$

where the objective is maximized, the constraints are equalities and the variables are all nonnegative.

This is done as follows:

- If the problem is *min* z, convert it to *max* $-z$.
- If a constraint is $a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n \leq b_i$, convert it into an equality constraint by adding a nonnegative *slack* variable s_i. The resulting constraint is $a_{i1}x_1+a_{i2}x_2+\ldots+a_{in}x_n+s_i = b_i$, where $s_i \geq 0$.
- If a constraint is $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n \geq b_i$, convert it into an equality constraint by subtracting a nonnegative *surplus* variable s_i. The resulting constraint is $a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n - s_i = b_i$, where $s_i \geq 0$.
- If some variable x_j is unrestricted in sign, replace it everywhere in the formulation by $x'_j - x''_j$, where $x'_j \geq 0$ and $x''_j \geq 0$.

Example 7.1.1 *Transform the following linear program into standard form.*

$$
\begin{array}{lcccl}
Min & -2x_1 & +3x_2 & & \\
 & x_1 & -3x_2 & +2x_3 & \leq 3 \\
 & -x_1 & +2x_2 & & \geq 2 \\
 & x_1 \ urs, & x_2 \geq 0, & x_3 \geq 0 &
\end{array}
$$

Let us first turn the objective into a *max* and the constraints into equalities.

$$
\begin{array}{lcccccl}
\text{Max} & 2x_1 & -3x_2 & & & & \\
 & x_1 & -3x_2 & +2x_3 & +s_1 & & = 3 \\
 & -x_1 & +2x_2 & & & -s_2 & = 2 \\
 & x_1 \text{ urs}, & x_2 \geq 0, & x_3 \geq 0 & s_1 \geq 0 & s_2 \geq 0 &
\end{array}
$$

The last step is to convert the unrestricted variable x_1 into two nonnegative variables: $x_1 = x_1' - x_1''$.

$$
\begin{array}{lccccccl}
\text{Max} & 2x_1' & -2x_1'' & -3x_2 & & & & \\
 & x_1' & -x_1'' & -3x_2 & +2x_3 & +s_1 & & = 3 \\
 & -x_1' & +x_1'' & +2x_2 & & & -s_2 & = 2 \\
 & x_1' \geq 0, & x_1'' \geq 0, & x_2 \geq 0, & x_3 \geq 0 & s_1 \geq 0 & s_2 \geq 0 &
\end{array}
$$

7.2 Solution of Linear Programs by the Simplex Method

For simplicity, in this course we solve "by hand" only the case where the constraints are of the form $\leq$ and the right-hand-sides are nonnegative. We will explain the steps of the simplex method while we progress through an example.

Example 7.2.1 *Solve the linear program*

$$
\begin{array}{lccl}
max & x_1 & +x_2 & \\
 & 2x_1 & +x_2 & \leq 4 \\
 & x_1 & +2x_2 & \leq 3 \\
 & x_1 \geq 0, & x_2 \geq 0 &
\end{array}
$$

First, we convert the problem into standard form by adding slack variables $x_3 \geq 0$ and $x_4 \geq 0$.

$$
\begin{array}{lccccl}
\text{max} & x_1 & +x_2 & & & \\
 & 2x_1 & +x_2 & +x_3 & & = 4 \\
 & x_1 & +2x_2 & & +x_4 & = 3 \\
 & x_1 \geq 0, & x_2 \geq 0 & x_3 \geq 0, & x_4 \geq 0 &
\end{array}
$$

Let z denote the objective function value. Here, $z = x_1 + x_2$ or, equivalently,

$$z - x_1 - x_2 = 0.$$

Putting this equation together with the constraints, we get the following system of linear equations.

$$
\begin{array}{cccccl l}
z & -x_1 & -x_2 & & & = 0 & \text{Row } 0 \\
 & 2x_1 & +x_2 & +x_3 & & = 4 & \text{Row } 1 \\
 & x_1 & +2x_2 & & +x_4 & = 3 & \text{Row } 2
\end{array} \tag{7.1}
$$

Our goal is to maximize z, while satisfying these equations and, in addition, $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, $x_4 \geq 0$.

Note that the equations are already in the form that we expect at the last step of the Gauss-Jordan procedure. Namely, the equations are solved in terms of the nonbasic variables x_1, x_2. The variables (other than the special variable z) which appear in only one equation are the *basic variables*. Here the basic variables are x_3 and x_4. A *basic solution* is obtained from the system of equations by setting the nonbasic variables to zero. Here this yields

$$x_1 = x_2 = 0 \quad x_3 = 4 \quad x_4 = 3 \quad z = 0.$$

Is this an optimal solution or can we increase z? (Our goal)

By looking at Row 0 above, we see that we can increase z by increasing x_1 or x_2. This is because these variables have a negative coefficient in Row 0. If all coefficients in Row 0 had been nonnegative, we could have concluded that the current basic solution is optimum, since there would be no way to increase z (remember that all variables x_i must remain ≥ 0). We have just discovered the first rule of the simplex method.

Rule 1 *If all variables have a nonnegative coefficient in Row 0, the current basic solution is optimal.*
Otherwise, pick a variable x_j with a negative coefficient in Row 0.

The variable chosen by Rule 1 is called the *entering* variable. Here let us choose, say, x_1 as our entering variable. It really does not matter which variable we choose as long as it has a negative coefficient in Row 0. The idea is to *pivot* in order to make the nonbasic variable x_1 become a basic variable. In the process, some basic variable will become nonbasic (the *leaving* variable). This *change of basis* is done using the Gauss-Jordan procedure. What is needed next is to choose the *pivot element*. It will be found using Rule 2 of the simplex method. In order to better understand the rationale behing this second rule, let us try both possible pivots and see why only one is acceptable.

First, try the pivot element in Row 1.

$$\begin{array}{llllllll}
z & -x_1 & -x_2 & & & = & 0 & \text{Row 0} \\
 & \mathbf{2x_1} & +x_2 & +x_3 & & = & 4 & \text{Row 1} \\
 & x_1 & +2x_2 & & +x_4 & = & 3 & \text{Row 2}
\end{array}$$

This yields

$$\begin{array}{llllllll}
z & & -\frac{1}{2}x_2 & +\frac{1}{2}x_3 & & = & 2 & \text{Row 0} \\
 & x_1 & +\frac{1}{2}x_2 & +\frac{1}{2}x_3 & & = & 2 & \text{Row 1} \\
 & & \frac{3}{2}x_2 & -\frac{1}{2}x_3 & +x_4 & = & 1 & \text{Row 2}
\end{array}$$

with basic solution $x_2 = x_3 = 0 \quad x_1 = 2 \quad x_4 = 1 \quad z = 2.$

Now, try the pivot element in Row 2.

$$\begin{array}{llllllll}
z & -x_1 & -x_2 & & & = & 0 & \text{Row 0} \\
 & 2x_1 & +x_2 & +x_3 & & = & 4 & \text{Row 1} \\
 & \mathbf{x_1} & +2x_2 & & +x_4 & = & 3 & \text{Row 2}
\end{array}$$

This yields

$$\begin{array}{rrrrcrl} z & & +x_2 & & +x_4 & = & 3 & \quad \text{Row } 0 \\ & & -3x_2 & +x_3 & -2x_4 & = & -2 & \quad \text{Row } 1 \\ & x_1 & +2x_2 & & +x_4 & = & 3 & \quad \text{Row } 2 \end{array}$$

with basic solution $x_2 = x_4 = 0 \quad x_1 = 3 \quad x_3 = -2 \quad z = 3$.

Which pivot should we choose? The first one, of course, since the second yields an *infeasible* basic solution! Indeed, remember that we must keep all variables ≥ 0. With the second pivot, we get $x_3 = -2$ which is infeasible. How could we have known this ahead of time, before actually performing the pivots? The answer is, by comparing the ratios $\frac{\text{Right Hand Side}}{\text{Entering Variable Coefficient}}$ in Rows 1 and 2 of (7.1). Here we get $\frac{4}{2}$ in Row 1 and $\frac{3}{1}$ in Row 2. If you pivot in a row with *minimum* ratio, you will end up with a feasible basic solution (i.e. you will not introduce negative entries in the Right Hand Side), whereas if you pivot in a row with a ratio which is not minimum you will always end up with an infeasible basic solution. Just simple algebra! A negative pivot element would not be good either, for the same reason. We have just discovered the second rule of the simplex method.

Rule 2 *For each Row i, $i \geq 1$, where there is a strictly positive "entering variable coefficient", compute the ratio of the Right Hand Side to the "entering variable coefficient". Choose the pivot row as being the one with MINIMUM ratio.*

Once you have idendified the pivot element by Rule 2, you perform a Gauss-Jordan pivot. This gives you a new basic solution. Is it an optimal solution? This question is addressed by Rule 1, so we have closed the loop. The simplex method iterates between Rules 1, 2 and pivoting until Rule 1 guarantees that the current basic solution is optimal. That's all there is to the simplex method.

After our first pivot, we obtained the following system of equations.

$$\begin{array}{rrrrrcrl} z & & -\frac{1}{2}x_2 & +\frac{1}{3}x_3 & & = & 2 & \quad \text{Row } 0 \\ & x_1 & +\frac{1}{2}x_2 & +\frac{1}{2}x_3 & & = & 2 & \quad \text{Row } 1 \\ & & \frac{3}{2}x_2 & -\frac{1}{2}x_3 & +x_4 & = & 1 & \quad \text{Row } 2 \end{array}$$

with basic solution $x_2 = x_3 = 0 \quad x_1 = 2 \quad x_4 = 1 \quad z = 2$.

Is this solution optimal? No, Rule 1 tells us to choose x_2 as entering variable. Where should we pivot? Rule 2 tells us to pivot in Row 2, since the ratios are $\frac{2}{1/2} = 4$ for Row 1, and $\frac{1}{3/2} = \frac{2}{3}$ for Row 2, and the minimum occurs in Row 2. So we pivot on $\frac{3}{2}x_2$ in the above system of equations. This yields

$$\begin{array}{rrrrrcrl} z & & & +\frac{1}{3}x_3 & +\frac{1}{3}x_4 & = & \frac{7}{3} & \quad \text{Row } 0 \\ & x_1 & & +\frac{2}{3}x_3 & -\frac{1}{3}x_4 & = & \frac{5}{3} & \quad \text{Row } 1 \\ & & x_2 & -\frac{1}{3}x_3 & +\frac{2}{3}x_4 & = & \frac{2}{3} & \quad \text{Row } 2 \end{array}$$

with basic solution $x_3 = x_4 = 0 \quad x_1 = \frac{5}{3} \quad x_2 = \frac{2}{3} \quad z = \frac{7}{3}$.

Now Rule 1 tells us that this basic solution is optimal, since there are no more negative entries in Row 0.

All the above computations can be represented very compactly in *tableau form*.

z	x_1	x_2	x_3	x_4	RHS		Basic solution	
1	-1	-1	0	0	0	basic	$x_3 = 4$	$x_4 = 3$
0	2	1	1	0	4	nonbasic	$x_1 = x_2 = 0$	
0	1	2	0	1	3	$z = 0$		
1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0	2	basic	$x_1 = 2$	$x_4 = 1$
0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2	nonbasic	$x_2 = x_3 = 0$	
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	1	$z = 2$		
1	0	0	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{7}{3}$	basic	$x_1 = \frac{5}{3}$	$x_2 = \frac{2}{3}$
0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$	nonbasic	$x_3 = x_4 = 0$	
0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$z = \frac{7}{3}$		

Since the above example has only two variables, it is interesting to interpret the steps of the simplex method graphically. See Figure 7.1. The simplex method starts in the corner point ($x_1 = 0, x_2 = 0$) with $z = 0$. Then it discovers that z can increase by increasing, say, x_1. Since we keep $x_2 = 0$, this means we move along the x_1 axis. How far can we go? Only until we hit a constraint: if we went any further, the solution would become infeasible. That's exactly what Rule 2 of the simplex method does: the minimum ratio rule identifies the first constraint that will be encountered. And when the constraint is reached, its slack x_3 becomes zero. So, after the first pivot, we are at the point ($x_1 = 2, x_2 = 0$). Rule 1 discovers that z can be increased by increasing x_2 while keeping $x_3 = 0$. This means that we move along the boundary of the feasible region $2x_1 + x_2 = 4$ until we reach another constraint! After pivoting, we reach the optimal point ($x_1 = \frac{5}{3}, x_2 = \frac{2}{3}$).

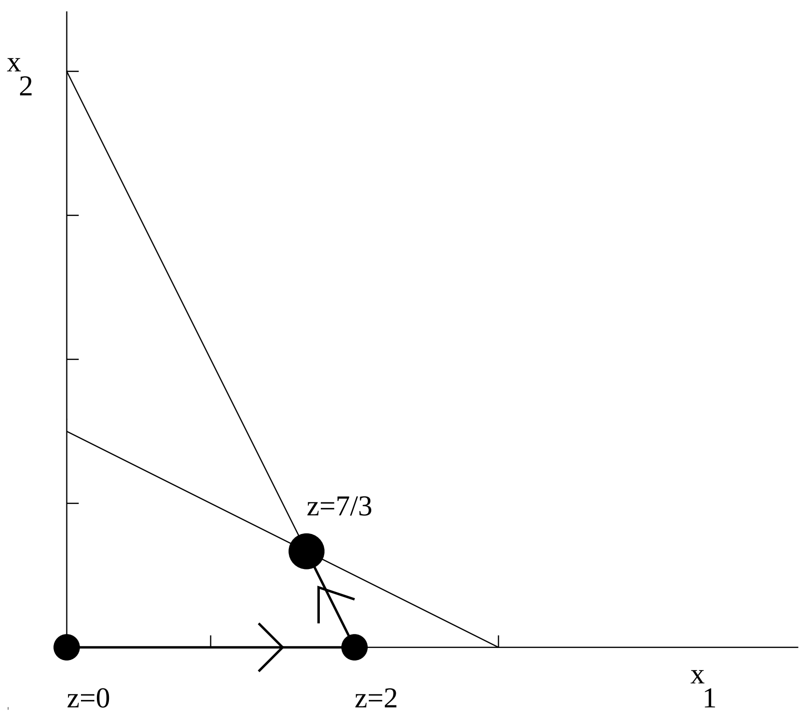

Figure 7.1: Graphical Interpretation

Exercise 62 Solve the following linear program by the simplex method.

$$\begin{array}{lrrrcl}
\max & 4x_1 & +x_2 & -x_3 & & \\
 & x_1 & & +3x_3 & \leq & 6 \\
 & 3x_1 & +x_2 & +3x_3 & \leq & 9 \\
 & x_1 \geq 0, & x_2 \geq 0 & x_3 \geq 0 & &
\end{array}$$

7.3 Alternate Optimal Solutions, Degeneracy, Unboudedness, Infeasibility

Alternate Optimal Solutions

Let us solve a small variation of the earlier example, with the same constraints but a slightly different objective:

$$\begin{array}{lrrcl}
\max & x_1 & +\frac{1}{2}x_2 & & \\
 & 2x_1 & +x_2 & \leq & 4 \\
 & x_1 & +2x_2 & \leq & 3 \\
 & x_1 \geq 0, & x_2 \geq 0 & &
\end{array}$$

As before, we add slacks x_3 and x_4, and we solve by the simplex method, using tableau representation.

z	x_1	x_2	x_3	x_4	RHS		Basic solution	
1	-1	$-\frac{1}{2}$	0	0	0	basic	$x_3 = 4$	$x_4 = 3$
0	2	1	1	0	4	nonbasic	$x_1 = x_2 = 0$	
0	1	2	0	1	3	$z = 0$		
1	0	0	$\frac{1}{2}$	0	2	basic	$x_1 = 2$	$x_4 = 1$
0	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2	nonbasic	$x_2 = x_3 = 0$	
0	0	$\frac{3}{2}$	$-\frac{1}{2}$	1	1	$z = 2$		

Now Rule 1 shows that this is an optimal solution. Interestingly, the coefficient of the nonbasic variable x_2 in Row 0 happens to be equal to 0. Going back to the rationale that allowed us to derive Rule 1, we observe that, if we increase x_2 (from its current value of 0), this will not effect the value of z. Increasing x_2 produces changes in the other variables, of course, through the equations in Rows 1 and 2. In fact, we can use Rule 2 and pivot to get a different basic solution with the same objective value $z = 2$.

z	x_1	x_2	x_3	x_4	RHS		Basic solution	
1	0	0	$\frac{1}{2}$	0	2	basic	$x_1 = \frac{5}{3}$	$x_2 = \frac{2}{3}$
0	1	0	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{5}{3}$	nonbasic	$x_3 = x_4 = 0$	
0	0	1	$-\frac{1}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$z = 2$		

Note that the coefficient of the nonbasic variable x_4 in Row 0 is equal to 0. Using x_4 as entering variable and pivoting, we would recover the previous solution!

Degeneracy

Example 7.3.1

$$
\begin{array}{lrrll}
max & 2x_1 & +x_2 & & \\
 & 3x_1 & +x_2 & \leq & 6 \\
 & x_1 & -x_2 & \leq & 2 \\
 & & x_2 & \leq & 3 \\
 & x_1 \geq 0, & x_2 \geq 0 & &
\end{array}
$$

Let us solve this problem using the –by now familiar– simplex method. In the initial tableau, we can choose x_1 as the entering variable (Rule 1) and Row 2 as the pivot row (the minimum ratio in Rule 2 is a tie, and ties are broken arbitrarily). We pivot and this yields the second tableau below.

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic solution
1	−2	−1	0	0	0	0	basic $x_3 = 6$ $x_4 = 2$ $x_5 = 3$
0	3	1	1	0	0	6	nonbasic $x_1 = x_2 = 0$
0	1	−1	0	1	0	2	$z = 0$
0	0	1	0	0	1	3	
1	0	−3	0	2	0	4	basic $x_1 = 2$ $x_3 = 0$ $x_5 = 3$
0	0	4	1	−3	0	0	nonbasic $x_2 = x_4 = 0$
0	1	−1	0	1	0	2	$z = 4$
0	0	1	0	0	1	3	

Note that this basic solution has a basic variable (namely x_3) which is equal to zero. When this occurs, we say that the basic solution is *degenerate*. Should this be of concern? Let us continue the steps of the simplex method. Rule 1 indicates that x_2 is the entering variable. Now let us apply Rule 2. The ratios to consider are $\frac{0}{4}$ in Row 1 and $\frac{3}{1}$ in Row 3. The minimum ratio occurs in Row 1, so let us perform the corresponding pivot.

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic solution
1	0	0	$\frac{3}{4}$	$-\frac{1}{4}$	0	4	basic $x_1 = 2$ $x_2 = 0$ $x_5 = 3$
0	0	1	$\frac{1}{4}$	$-\frac{3}{4}$	0	0	nonbasic $x_3 = x_4 = 0$
0	1	0	$\frac{1}{4}$	$\frac{1}{4}$	0	2	$z = 4$
0	0	0	$-\frac{1}{4}$	$\frac{3}{4}$	1	3	

We get exactly the same solution! The only difference is that we have interchanged the names of a nonbasic variable with that of a degenerate basic variable (x_2 and x_3). Rule 1 tells us the solution is not optimal, so let us continue the steps of the simplex method. Variable x_4 is the entering variable and the last row wins the minimum ratio test. After pivoting, we get the tableau:

z	x_1	x_2	x_3	x_4	x_5	RHS	Basic solution
1	0	0	$\frac{2}{3}$	0	$\frac{1}{3}$	5	basic $x_1 = 1$ $x_2 = 3$ $x_4 = 4$
0	0	1	0	0	1	3	nonbasic $x_3 = x_5 = 0$
0	1	0	$\frac{1}{3}$	0	$-\frac{1}{3}$	1	$z = 5$
0	0	0	$-\frac{1}{3}$	1	$\frac{4}{3}$	4	

By Rule 1, this is the optimal solution. So, after all, degeneracy did not prevent the simplex method to find the optimal solution in this example. It just slowed things down a little. Unfortunately, on other examples, degeneracy may lead to *cycling*, i.e. a sequence of pivots that goes through the same tableaus and repeats itself indefinitely. In theory, cycling can be avoided by choosing the entering variable with smallest index in Rule 1, among all those with a negative coefficient in Row 0, and by breaking ties in the minimum ratio test by choosing the leaving variable with smallest index (this is known as Bland's rule). This rule, although it guaranties that cycling will never occur, turns out to be somewhat inefficient. Actually, in commercial codes, no effort is made to avoid cycling. This may come as a surprise, since degeneracy is a frequent occurence. But there are two reasons for this:

- Although degeneracy is frequent, cycling is extremely rare.
- The precision of computer arithmetic takes care of cycling by itself: round off errors accumulate and eventually gets the method out of cycling.

Our example of degeneracy is a 2-variable problem, so you might want to draw the constraint set in the plane and interpret degeneracy graphically.

Unbounded Optimum

Example 7.3.2

$$\begin{array}{lrrcl} max & 2x_1 & +x_2 & & \\ & -x_1 & +x_2 & \leq & 1 \\ & x_1 & -2x_2 & \leq & 2 \\ & x_1 \geq 0, & x_2 \geq 0 & & \end{array}$$

Solving by the simplex method, we get:

z	x_1	x_2	x_3	x_4	RHS		Basic solution	
1	−2	−1	0	0	0	basic	$x_3 = 1$	$x_4 = 2$
0	−1	1	1	0	1	nonbasic	$x_1 = x_2 = 0$	
0	1	−2	0	1	2	$z = 0$		
1	0	−5	0	2	4	basic	$x_1 = 2$	$x_3 = 3$
0	0	−1	1	1	3	nonbasic	$x_2 = x_4 = 0$	
0	1	−2	0	1	2	$z = 4$		

At this stage, Rule 1 chooses x_2 as the entering variable, but there is no ratio to compute, since there is no positive entry in the column of x_2. As we start increasing x_2, the value of z increases (from Row 0) and the values of the basic variables increase as well (from Rows 1 and 2). There is nothing to stop them going off to infinity. So the problem is unbounded.

Interpret the steps of the simplex method graphically for this example.

Infeasible Linear Programs

It is easy to construct constraints that have no solution. The simplex method is able to identify such cases. We do not discuss it here. The interested reader is referred to Winston, for example.

Properties of Linear Programs

There are three possible outcomes for a linear program: it is infeasible, it has an unbounded optimum or it has an optimal solution.

If there is an optimal solution, there is a *basic* optimal solution. Remember that the number of basic variables in a basic solution is equal to the number of constraints of the problem, say m. So, even if the total number of variables, say n, is greater than m, at most m of these variables can have a positive value in an optimal basic solution.

Exercise 63 The following tableaus were obtained in the course of solving linear programs with 2 nonnegative variables x_1 and x_2 and 2 inequality constraints (the objective function z is maximized). Slack variables s_1 and s_2 were added. In each case, indicate whether the linear program

(i) is unbounded

(ii) has a unique optimum solution

(iii) has an alternate optimum solution

(iv) is degenerate (in this case, indicate whether any of the above holds).

(a)

z	x_1	x_2	s_1	s_2	RHS
1	0	3	2	0	20
0	1	-2	-1	0	4
0	0	-1	0	1	2

(b)

z	x_1	x_2	s_1	s_2	RHS
1	0	-1	0	2	20
0	0	0	1	-2	5
0	1	-2	0	3	6

(c)

z	x_1	x_2	s_1	s_2	RHS
1	2	0	0	1	8
0	3	1	0	-2	4
0	-2	0	1	1	0

(d)

z	x_1	x_2	s_1	s_2	RHS
1	0	0	2	0	5
0	0	-1	1	1	4
0	1	1	-1	0	4

Exercise 64 Suppose the following tableau was obtained in the course of solving a linear program with nonnegative variables x_1, x_2, x_3 and two inequalities. The objective function is maximized and slack variables s_1 and s_2 were added.

z	x_1	x_2	x_3	s_1	s_2	RHS
1	0	a	b	0	4	82
0	0	-2	2	1	3	c
0	1	-1	3	0	-5	3

Give conditions on a, b and c that are required for the following statements to be true:

(i) The current basic solution is a feasible basic solution.

Assume that the condition found in (i) holds in the rest of the exercise.

(ii) The current basic solution is optimal.

(iii) The linear program is unbounded (for this question, assume that $b > 0$).

(iv) The currrent basic solution is optimal and there are alternate optimal solutions (for this question, assume $a > 0$).

Exercise 65 A plant can manufacture five products P_1, P_2, P_3, P_4 and P_5. The plant consists of two work areas: the job shop area A_1 and the assembly area A_2. The time required to process one unit of product P_j in work area A_i is p_{ij} (in hours), for $i = 1, 2$ and $j = 1, \ldots, 5$. The weekly capacity of work area A_i is C_i (in hours). The company can sell all it produces of product P_j at a profit of s_j, for $i = 1, \ldots, 5$.

The plant manager thought of writing a linear program to maximize profits, but never actually did for the following reason: From past experience, he observed that the plant operates best when at most two products are manufactured at a time. He believes that if he uses linear programming, the optimal solution will consist of producing all five products. Do you agree with him? Explain, based on your knowledge of linear programming.

Exercise 66 Consider the linear program

Maximize $5x_1 + 3x_2 + x_3$

Subject to

$x_1 + x_2 + x_3 \leq 6$

$5x_1 + 3x_2 + 6x_3 \leq 15$

$x_1, x_2, x_3 \geq 0$

and an associated tableau

z	x_1	x_2	x_3	s_1	s_2	RHS
1	0	0	5	0	1	15
0	0	.4	-0.2	1	-.2	3
0	1	.6	1.2	0	.2	3

(a) What basic solution does this tableau represent? Is this solution optimal? Why or why not?

(b) Does this tableau represent a unique optimum. If not, find an alternative optimal solution.

Answer:

(a) *The solution is $x_1 = 3$, $x_2, x_3 = 0$, objective 15. It is optimal since cost row is all at least 0.*

(b) *It is not unique (since x_2 has reduced cost 0 but is not basic). Alternative found by pivoting in x_2 (question could have asked for details on such a pivot) for solution $x_2 = 5$, $x_1, x_3 = 0$, objective 15.*