



TOPIC : 1.6 Moments-Moment Generating function

Moments :

Mathematical Expectation :

Let x be a random variable with pdf (or pmf) $f(x)$. Then its mathematical expectation, denoted by $E(x)$ is given by

$$E(x) = \int_{-\infty}^{\infty} x f(x) dx \quad \text{[for continuous R.V.]}$$

$$E(x) = \sum x f(x) \quad \text{[for discrete R.V.]}$$

Moments: [Discrete case]

Let x be discrete R.V. taking values x_1, x_2, \dots with pmf $p(x)$ then r -th moment about the origin is

$$\mu_r' \text{ (about the origin) } = \sum x^r p(x)$$

(or) raw moments

and μ_r' (about any point $x=A$) = $\sum (x-A)^r p(x)$

& μ_r' (about mean) } = $\sum (x - \text{mean})^r p(x)$
(or) [central moments] } [mean = \bar{x}]

Moments [continuous case]

If x is a continuous RV with pdf $f(x)$ then defined in the interval (a, b)

$$\mu_r' \text{ (about origin) } = \int_a^b x^r f(x) dx$$

(or) raw moments

$$\mu_r' \text{ (about a point A) } = \int_a^b (x-A)^r f(x) dx$$

$$\mu_r' \text{ (about the mean) } = \int_a^b (x - \bar{x})^r f(x) dx$$

(or) (central moments)



Relation between μ_r^* & μ_r

$$\mu_1^* = \text{mean } \bar{x}$$

$$\mu_2^* = \mu_2' - (\mu_1')^2 \quad [\text{variance}]$$

$$\mu_3^* = \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$$

$$\mu_4^* = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$$

Note:

(1) $\mu_1 = 0$ (always)

(2) $E[ax+b] = aE(x) + b$

(3) $E[\varphi(x)+a] = E[\varphi(x)] + a$

(4) $\text{var}(ax+b) = a^2 \text{var}(x)$

(5) $\text{var}(x+k) = \text{var}(x)$

$$E(ax+b) = \frac{\sum(ax+b)}{n} \\ = a \frac{\sum x}{n} + \frac{\sum b}{n} \\ = aE(x) + b$$



Problems based on Moments [Discrete Case]

The monthly demand for Allwyn watches is known to have the following Probability distribution.

Demand	1	2	3	4	5	6	7	8
Probability	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Find the expected demand for watches. Also compute the variance.

Solu:

x_i	1	2	3	4	5	6	7	8
$P(x_i)$	0.08	0.12	0.19	0.24	0.16	0.10	0.07	0.04

Let x be the R.V denoting the monthly demand for Allwyn watches.

$$E(x) = \sum_{i=1}^8 x_i p(x_i)$$
$$= 1(0.08) + 2(0.12) + 3(0.19) + 4(0.24) + 5(0.16) + 6(0.10) + 7(0.07) + 8(0.04)$$
$$= 4.06$$
$$E(x^2) = \sum_{i=1}^8 x_i^2 p(x_i)$$
$$= 1(0.08) + 4(0.12) + 9(0.19) + 16(0.24) + 25(0.16) + 36(0.10) + 49(0.07) + 64(0.04)$$
$$= 19.7$$
$$\text{Var}(x) = E(x^2) - [E(x)]^2$$
$$= 19.7 - (4.06)^2$$
$$= 19.7 - 16.48$$
$$= 3.22$$



Problem based on Moments [Continuous Case]

1. The density function of a random variable 'x' is given by $f(x) = kx(2-x)$, $0 \leq x \leq 2$. Find k, mean, variance and rth moment.

Solu: Given $f(x) = kx(2-x)$, $0 \leq x \leq 2$
To find k:

$$\int f(x) dx = 1$$

$$\int_0^2 kx(2-x) dx = 1$$

$$k \int_0^2 (2x - x^2) dx = 1$$

$$k \left[\frac{2x^2}{2} - \frac{x^3}{3} \right]_0^2 = 1$$

$$k \left[4 - \frac{8}{3} - (0-0) \right] = 1$$

$$k \left[\frac{12-8}{3} \right] = 1$$

$$\Rightarrow k = \frac{3}{4}$$

To find rth moment: $E(x^r)$

$$E(x^r) = \int_0^2 x^r f(x) dx$$

$$= \frac{3}{4} \int_0^2 x^r (2x - x^2) dx$$

$$= \frac{3}{4} \int_0^2 (2x^{r+1} - x^{r+2}) dx$$

$$= \frac{3}{4} \left[\frac{2x^{r+2}}{r+2} - \frac{x^{r+3}}{r+3} \right]_0^2$$

$$= \frac{3}{4} \left[2 \frac{2^{r+2}}{r+2} - \frac{2^{r+3}}{r+3} - (0-0) \right]$$



$$\begin{aligned} &= \frac{3}{4} \int \left[\frac{2^{x+3}}{x+2} - \frac{2^{x+3}}{x+3} \right] \\ &= \frac{3}{4} \cdot 2^{x+3} \left[\frac{1}{x+2} - \frac{1}{x+3} \right] \\ &= \frac{3}{4} \cdot 2^x \cdot 2^3 \left[\frac{x+3 - x - 2}{(x+2)(x+3)} \right] \\ &= \frac{3}{4} \cdot 2^x \cdot 8 \left[\frac{1}{(x+2)(x+3)} \right] \end{aligned}$$

$$E(x^2) = \frac{6 \cdot 2^x}{(x+2)(x+3)} \quad \text{--- (1)}$$

To find mean $E(x)$ & $\text{var}(x)$

$$\text{put } x=1 \text{ in (1) } E(x) = \frac{6 \cdot 2}{3 \cdot 4} \\ = 1$$

$$\text{put } x=2 \text{ in (1) } E(x^2) = \frac{6 \cdot 2^2}{4 \cdot 5} \\ = \frac{6}{5}$$

$$\therefore \text{Mean } E(x) = 1$$

$$\begin{aligned} \text{var}(x) &= E(x^2) - [E(x)]^2 \\ &= \frac{6}{5} - 1 \\ &= \frac{6-5}{5} = \frac{1}{5} // \end{aligned}$$

Moment Generating function (MGF)

Moment generating function of a random Variable x about the origin is defined as

$$M_x(t) = E[e^{tx}] = \sum e^{tx} p(x), \text{ if } x \text{ is discrete}$$

$$= \int_{-\infty}^{\infty} e^{tx} f(x) dx, \text{ if } x \text{ is continuous.}$$

By MGF about mean μ is defined as

$$M_{x-\mu}(t) = E[e^{t(x-\mu)}] = \sum e^{t(x-\mu)} p(x), \text{ if } x \text{ is discrete}$$

$$= \int e^{t(x-\mu)} f(x) dx \text{ if } x \text{ is contin}$$

1. Prove that the r^{th} moment of the R.V x about origin $M_x(t) = \sum \frac{t^r}{r!} \mu_r$.

Proof:
we know that
 $M_x(t) = E[e^{tx}]$

$$= E\left[1 + \frac{tx}{1!} + \frac{(tx)^2}{2!} + \frac{(tx)^3}{3!} + \dots + \frac{(tx)^r}{r!} + \dots\right]$$

[formula $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$]

$$= E\left[1 + \frac{tx}{1!} + \frac{t^2 x^2}{2!} + \dots + \frac{t^r x^r}{r!} + \dots\right]$$

$$= 1 + t E[x] + \frac{t^2}{2!} E[x^2] + \dots + \frac{t^r}{r!} E[x^r] + \dots$$

$$M_x(t) = 1 + t \mu_1 + \frac{t^2}{2!} \mu_2 + \dots + \frac{t^r}{r!} \mu_r + \dots$$

$$M_x(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \mu_r$$

note: \therefore Thus r^{th} moment = coefficient of $\frac{t^r}{r!}$.



Note: 2 $M_x' = \frac{d^r}{dt^r} [M_x(t)]_{t=0}$.

Properties of MGF:

1. $M_{X+a}(t) = e^{-at} M_x(t)$ $M_x(t) = E[e^{tx}]$
 $M_{X+a}(t) = E[e^{t(x+a)}]$

Proof: $M_{X+a}(t) = E[e^{t(x+a)}]$
 $= e^{at} E[e^{tx}]$
 $= e^{at} M_x(t)$

2. Let $Y = ax+b$, where X is a RV with moment generating fun. $M_X(t)$. Then $M_Y(t) = e^{bt} M_X(at)$

Proof: $M_Y(t) = E[e^{tY}]$
 $= E[e^{t(ax+b)}]$
 $= e^{bt} E[e^{tax}]$
 $= e^{bt} M_X(at)$

3. If $M_X(t) = E[e^{tx}]$ then $M_{cX}(t) = M_X(ct)$

Solu: $M_{cX}(t) = E[e^{t(cX)}] = E[e^{(ct)X}] = M_X(ct)$

4. If X & Y are two independent RV, then $M_{X+Y}(t) = M_X(t) \cdot M_Y(t)$

Proof: $M_{X+Y}(t) = E[e^{t(X+Y)}]$
 $= E[e^{tX+tY}]$ $E[XY] = E[X] \cdot E[Y]$
 \downarrow
 X, Y
 $= E[e^{tX} \cdot e^{tY}]$ $\therefore X, Y$ are independent
 $= E[e^{tX}] \cdot E[e^{tY}]$

Note: If X_1, X_2, \dots, X_n are n independent RVs then
 $M_{X_1+X_2+\dots+X_n}(t) = M_{X_1}(t) \cdot M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t)$ (21)

Problems based on MGF [Discrete case]

1. A random variate x has probability function $P(x) = \frac{1}{2^x}; x=1,2,3, \dots$. Find the MGF, Mean & Variance.

Solu:

w.k.t $M_x(t) = \sum e^{tx} P(x)$ (for discrete P.V.)

$$= \sum e^{tx} \frac{1}{2^x}$$

$$= \sum \left(\frac{e^t}{2}\right)^x$$

$$= \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \left(\frac{e^t}{2}\right)^3 + \dots$$

$$= \frac{e^t}{2} \left\{ 1 + \frac{e^t}{2} + \left(\frac{e^t}{2}\right)^2 + \dots \right\}$$

$$\therefore M_x(t) = \frac{e^t}{2} \cdot \frac{1}{1 - \frac{e^t}{2}} = \frac{e^t}{2 - e^t}$$

$$M_1' = M_x'(0) = \left. \frac{d}{dt} \left[\frac{e^t}{2 - e^t} \right] \right|_{t=0}$$

$$= \left. \left[\frac{(2 - e^t)e^t - e^t(-e^t)}{(2 - e^t)^2} \right] \right|_{t=0}$$

$$= \left. \left[\frac{2e^t - e^{2t} + e^{2t}}{(2 - e^t)^2} \right] \right|_{t=0}$$

$$M_1' = 2$$

$$M_2' = \left. \frac{d^2}{dt^2} [M_x(t)] \right|_{t=0}$$

$$= \frac{(2 - e^t)^2(2e^t) - 4e^t(2 - e^t)(-e^t)}{(2 - e^t)^4}$$



$$= \left[\frac{(a-e^t)ae^t + He^{2t}}{(a-e^t)^3} \right]_{t=0}$$
$$= a+a = 6 //$$
$$\therefore \text{Var} = \mu_2' - (\mu_1')^2$$
$$= 6 - 4 = 2$$
$$\therefore \text{Mean} = 2 \quad \& \quad \text{Variance} = 2$$

2. Find the MGF for the distribution with

$$p(x) = \begin{cases} 2/3 & \text{at } x=1 \\ 1/3 & \text{at } x=2 \\ 0 & \text{otherwise} \end{cases} \text{ Also find mean \& \text{Variance.}$$

Sol: x takes the values 1 & 2 with probabilities $2/3$ & $1/3$ respectively.

$$M_x(t) = \sum_{x=1}^{\infty} e^{tx} p(x)$$
$$= \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$
$$\therefore M_x(t) = \frac{2}{3} e^t + \frac{1}{3} e^{2t}$$
$$M_x'(t) = \frac{2}{3} e^t + \frac{2}{3} e^{2t} \quad \text{--- (1)}$$
$$M_x''(t) = \frac{2}{3} e^t + \frac{4}{3} e^{2t} \quad \text{--- (2)}$$
$$\textcircled{1} \Rightarrow \mu_1' = M_x'(t) \Big|_{t=0} = 4/3$$
$$\textcircled{2} \Rightarrow \mu_2' = M_x''(t) \Big|_{t=0} = 6/3 = 2$$

Hence mean $\mu_1' = 4/3$ &

$$\text{Variance} = \mu_2' - \mu_1'^2$$
$$= 2 - 16/9 = 2/9 //$$



Problem based on MGF (continuous case)

1. Find the MGF of a random variable having the pdf $f(x) = \begin{cases} 1/3, & -1 < x < 2 \\ 0, & \text{otherwise} \end{cases}$.

Soln:

$$M_X(t) = E[e^{tx}] = \int e^{tx} f(x) dx$$
$$= \int_{-1}^2 e^{tx} \frac{1}{3} dx = \frac{1}{3} \left[\frac{e^{tx}}{t} \right]_{-1}^2$$
$$= \frac{1}{3t} (e^{2t} - e^{-t}), t \neq 0$$

For $t=0$ $M_X(0) = \lim_{t \rightarrow 0} \frac{e^{2t} - e^{-t}}{3t} = \frac{0}{0}$

$$= \lim_{t \rightarrow 0} \frac{2e^{2t} + e^{-t}}{3} = \frac{3}{3} = 1$$

$\therefore M_X(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t}, & t \neq 0 \\ 1, & t = 0 \end{cases}$

2. Find the MGF of a r.v x whose pdf is defined by $f(x) = \begin{cases} x, & \text{for } 0 \leq x < 1 \\ 2-x, & \text{for } 1 \leq x \leq 2 \\ 0, & \text{otherwise} \end{cases}$ hence find mean & var. dx

Soln:

$$M_X(t) = \int e^{tx} f(x) dx$$
$$= \int_0^1 2e^{tx} dx + \int_1^2 (2-x)e^{tx} dx$$
$$= \left[x \left[\frac{e^{tx}}{t} \right] - \frac{e^{tx}}{t^2} \right]_0^1 + \left[(2-x) \frac{e^{tx}}{t} + \frac{e^{tx}}{t^2} \right]_1^2$$
$$= \left[\frac{e^t}{t} - \frac{e^t}{t^2} - 0 + \frac{1}{t^2} \right] + \left[0 + \frac{e^{2t}}{t} - \frac{e^t}{t} - \frac{e^t}{t^2} \right]$$
$$= \left[1 - \frac{2e^t + e^{2t}}{t^2} \right] = \left(\frac{1-e^t}{t} \right)^2$$



to find mean & variance.

$$M_x(t) = \frac{[1 - e^{-t}]^2}{t^2} = \left[\frac{1 - e^{-t}}{t} \right]^2 \quad 2 \quad e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \dots$$
$$= \left[1 - \left(1 - \frac{t}{1} + \frac{t^2}{2} - \dots \right) \right]^2$$
$$= \left[\frac{t - \frac{t^2}{2} + \frac{t^3}{6} - \dots}{t} \right]^2$$
$$M_x'(t) = \left[1 - \frac{t}{2} + \frac{t^2}{6} - \dots \right]$$
$$\mu_1' = \left[\frac{d}{dt} M_x(t) \right]_{t=0} = M_x'(0)$$
$$M_x'(1) = 2 \left(1 - \frac{t}{2} + \frac{t^2}{6} - \dots \right) \left(-\frac{1}{2} + \frac{t}{3} - \dots \right) \quad \text{--- (1)}$$
$$M_x'(0) = 2(1) \left(\frac{1}{3} \right) \Rightarrow \boxed{\text{Mean} = 1}$$
$$\textcircled{2} M_x''(t) = 2 \left(1 - \frac{t}{2} + \frac{t^2}{6} - \dots \right) \left(-\frac{1}{3} + \frac{t}{3} - \dots \right) + 2 \left(-\frac{1}{2} + \frac{t}{3} - \dots \right) \left(1 - \frac{t}{2} + \frac{t^2}{6} - \dots \right)$$
$$M_x''(0) = 2 \left(1 \right) \left(\frac{1}{3} \right) + 2 \left(\frac{1}{3} \right) \cdot (1)$$
$$= \frac{2}{3} + \frac{2}{3} = \frac{4}{3} = \frac{7}{3}$$

Variance of $x = E[x^2] - (E[x])^2$

$$= \mu_2' - \mu_1'^2$$
$$= \frac{7}{3} - 1 = \frac{4}{3}$$

Hence $\boxed{\text{Mean} = 1}$ & $\boxed{\text{Variance} = \frac{4}{3}}$