**TOPIC 2.11-** Transformation of random variable

## Transforms of two dimensional random variables

**Step-1** Find the joint density function of $(x, y)$ if it is not given.

**Step-2** Consider the new random variables $U$ and $V$ try to express the relation in the form $x = g_1(u,v)$, $y = g_2(u,v)$

**Step-3** Find the Jacobian of $(x,y)$ w.r.t $(u,v)$

i.e) $\frac{\partial(x,y)}{\partial(u,v)}$.

**Step-4** Write the formula of the pdf of

i.e) $f_{UV}(u,v) = f_{XY}(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right|$

**Step-5** Find the values of $f_U(u)$ and $f_V(v)$ using the method of finding the marginal densities.

(1) Two random variables $X$ and $Y$ have the following joint probability density function

$$f(x,y) = \begin{cases} x+y, & 0 \le x, y \le 1 \\ 0, & \text{otherwise} \end{cases}$$

Find the probability density function of the random variable $U = XY$.

<u>Step-1</u> The joint pdf of $(X, Y)$ is given by

$$f_{XY}(x,y) = x+y, \quad 0 \le x, y \le 1$$

<u>Step-2</u> Introducing the new r.v's

Given $U = XY$ and let $V = Y$

<u>Step-3</u> Expressing the above relation as $x = g_1(u,v)$ and $y = g_2(u,v)$

$$\frac{u}{y} \Rightarrow \boxed{x = \frac{u}{v}} \quad \text{and} \quad \boxed{y = v}$$

<u>Step-4</u> To find $|J|$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & 0 \\ -\frac{u}{v^2} & 1 \end{vmatrix}$$

$$= \frac{1}{v}$$

<u>Step-5</u> To find pdf of $(U, V)$

$$f_{UV}(u,v) = f(x,y)\,|J|$$

$$= \frac{1}{v}(x+y)$$

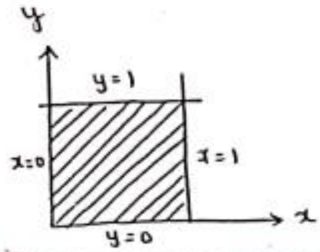
$$= \frac{1}{v}\left[\frac{u}{v} + v\right]$$

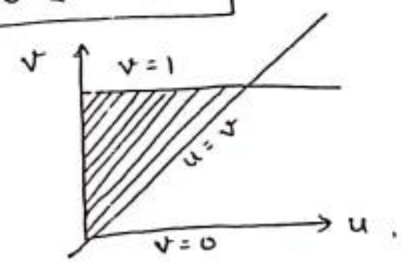
Step-6 Changing $(x, y)$ domain into $(u, v)$ domain

Since $0 \le y \le 1 \Rightarrow \boxed{0 \le v \le 1}$

$0 \le x \le 1 \Rightarrow 0 \le \frac{u}{v} \le 1$

$\Rightarrow \boxed{0 \le u \le v}$




Step-7 To find the pdf of $U = XY$

$$f(u) = \int_{-\infty}^{\infty} f(u,v)\,dv = \int_{u}^{1} \frac{1}{v}\left[\frac{u}{v}+v\right] dv$$

$$= \int_{u}^{1} \left[\frac{u}{v^2}+1\right] dv = \left[u\,\frac{v^{-1}}{-1}+v\right]_{u}^{1}$$

$$= \left[(1-u) - \left(u - \frac{u}{u}\right)\right] = \left[1-u-u+1\right]$$

$$= 2 - 2u$$

$$f_U(u) = 2(1-u), \quad 0 < u < 1$$

(2) If x and y are independent random variables with density function $f(x) = \begin{cases} 1, & 1 \le x \le 2 \\ 0, & \text{otherwise} \end{cases}$ and $f(y) = \begin{cases} y/6 ; & 2 \le y \le 4 \\ 0 ; & \text{otherwise} \end{cases}$ find the density function of $U = XY$.

Step-1 To find the joint pdf of $(x, y)$

Given x and y are independent.

$\therefore$ i.e) $f(x, y) = f(x)\ f(y)$

$$f(x, y) = \frac{y}{6}, \quad 1 \le x \le 2, \quad 2 \le y \le 4$$

Step-2 Introducing the new r.v's

Given $U = XY$ and let $V = Y$

Step-3 Expressing the above relation as $x = g_1(u, v)$ and $y = g_2(u, v)$

$$x = \frac{U}{y} \Rightarrow \boxed{x = \frac{u}{v}}$$

$$\boxed{y = v}$$

step-4 To find $|J|$

$$|J| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{v} & 0 \\ -\frac{u}{v^2} & 1 \end{vmatrix} = \frac{1}{v}$$

step-5 To find pdf of $(u,v)$

$$f_{UV}(u,v) = f_{XY}(x,y)\,|J| = (x/y)\cdot \frac{1}{v}$$

$$= \frac{1}{y}\left(\frac{u}{v}\cdot v\right) = \frac{u}{v}\cdot y$$

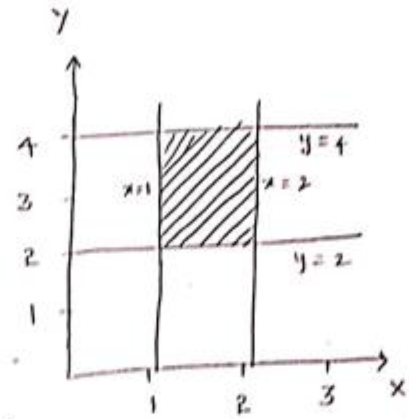
$$= \frac{y}{6}\cdot\frac{1}{v} = \frac{v}{6}\cdot\frac{1}{v} = \frac{1}{6}$$

step-6 changing $(x,y)$ domain into $(u,v)$

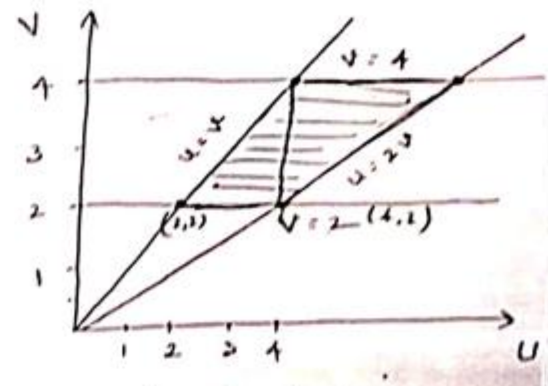
since $2 \le y \le 4 \Rightarrow \boxed{2 \le v \le 4}$

$1 \le x \le 2 \Rightarrow 1 \le \frac{u}{v} \le 2$

$\Rightarrow \boxed{v \le u \le 2v}$




(x, y) - domain




(u, v) - domain

Step-7 To find the pdf of $U = XY$.

In $2 \le u \le 4$, $f(u) = \int_{2}^{u} f(u,v)\, dv$

$$= \int_{2}^{u} \frac{1}{6}\, dv = \frac{1}{6}(u-2)$$

In $4 \le u \le 8$, $f(u) = \int_{u/2}^{4} f(u,v)\, dv$

$$= \int_{u/2}^{4} \frac{1}{6}\, dv = \frac{1}{6}\left(4 - \frac{u}{2}\right) = \frac{1}{12}(8-u)$$

Given the joint density function of $x$ and $y$ as

$$f(x,y) = \begin{cases} \frac{1}{2} x e^{-y} & ; 0 < x < 2,\ y > 0 \\ 0 & ; \text{otherwise} \end{cases}$$

# Find the distribution of $X+Y$

**Step-1** The joint distribution fn. is given by $f(x,y) = \begin{cases} \frac{1}{2} x e^{-y}, & 0<x<2,\ y>0 \\ 0, & \text{Otherwise} \end{cases}$

**Step-2** Introducing the new r.v's

Let $U = X+Y$ and $V = Y$

**Step-3** $x = u-y$ | $y = v$

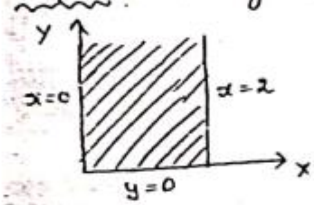
$x = u-v$

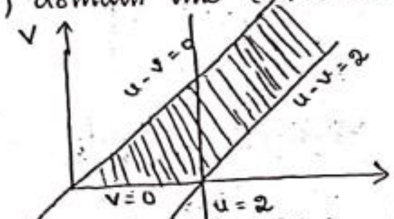
**Step-4** To find $|J|$ $\quad |J| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -1 & 1 \end{vmatrix} = 1$

**Step-5** To find the pdf of $U$ & $V$ $\quad f(u,v) = |J|\, f(x,y)$

$$f(u,v) = \frac{1}{2} x e^{-y} = \frac{1}{2}(u-v).e^{-v} = \frac{1}{2}\left[u e^{-v} - v e^{-v}\right]$$

**Step-6** change $(x,y)$ domain into $(u,v)$ domain




$0<x<2 \Rightarrow 0<u-v<2$

$y>0 \Rightarrow \boxed{v>0}$

$0<u<v+2$

**Step-7** To find the pdf of $U$, $\quad f(u) = \int_0^u f(u,v)\,dv + \int_{u-2}^u f(u,v)\,dv$

$$f(u) = \begin{cases} \frac{1}{2}\left(u+e^{-u}-1\right), & 0<u\le 2 \\ \frac{1}{2} e^{-u}\left(1+e^{2}\right), & 2<u<\infty \end{cases}$$