TOPIC : 2.2 Joint distributions Discrete Random variables

Joint Probability Function or Joint Probability Mass Function of the Discrete r.v X and Y

For two discrete random variables X and Y, we write the probability that X will take the value x_i and Y will take the value y_j as $P[X = x_i, Y = y_j]$. The function

$P[X = x_i, Y = y_j] = P(x_i, y_j) = P_{ij}$ is called the Joint probability function for discrete random variables X and Y. P_{ij} should satisfies the following conditions

(i) $P_{ij} \geq 0 \ \forall \ i, j$

(ii) $\sum_j \sum_i P_{ij} = 1$

Joint Probability Distribution of (X, Y)

The set of triplets $\{x_i, y_j, p_{ij}\}$, $i = 1, 2, \dots n$, $j = 1, 2, \dots m$ is called the joint probability distribution of (X, Y) and it can be represented in the form of table as given below.

X \ Y	y_1	y_2	----	y_m	$P(X = x_i)$
x_1	p_{11}	p_{12}	----	p_{1m}	$p_{1.}$
x_2	p_{21}	p_{22}	----	p_{2m}	$p_{2.}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
x_n	p_{n1}	p_{n2}	----	p_{nm}	$p_{n.}$
$P(Y = y_j)$	$p_{.1}$	$p_{.2}$	----	$p_{.m}$	1

Marginal probability function of X

If the joint probability distribution of two random variables X and Y is given, then the marginal probability function of X is given by

$$P_X(x_i) = P(X = x_i)$$
$$= P_{i1} + P_{i2} + \cdots + P_{im}$$
$$= \sum_{j=1}^{m} P_{ij} = P_{i.}$$

Marginal probability function of Y

If the joint probability distribution of two random variables X and Y is given, then the marginal probability function of Y is given by

$$P_Y(y_j) = P[Y = y_j)$$
$$= P_{1j} + P_{2j} + \cdots + P_{nj}$$
$$= \sum_{i=1}^{n} P_{ij} = P_{.j}$$

Conditional Probabilities

The conditional probability function of X given $Y = y_j$ is given by

$$P[X = x_i / Y = y_j] = \frac{P[X = x_i, Y = y_j]}{P[Y = y_j]} = \frac{P_{ij}}{P_{\cdot j}}$$

The conditional probability function of Y given $X = x_i$ is given by

$$P[Y = y_j / X = x_i] = \frac{P[Y = y_j, X = x_i]}{P[X = x_i]} = \frac{P_{ij}}{P_{i\cdot}}$$

1) The joint probability mass function of a two dimensional random variable (X, Y) is given by $P(x, y) = K(2x + y)$; $x = 1, 2$ and $y = 1, 2$ where K is constant. Find the value of K.

Given $P(x,y) = K(2x+y)$; $x = 1, 2$; $y = 1, 2$

X \ Y	1	2	$P(X=x)$
1	3K	4K	7K
2	5K	6K	11K
$P(Y=y)$	8K	10K	18K

$$\sum_j \sum_i P(x,y) = 1 \Rightarrow 18K = 1$$

$$\boxed{K = 1/18}$$

② For the bivariate probability distribution of (x,y) given below

X \ Y	1	2	3	4	5	6
0	0	0	1/32	2/32	2/32	3/32
1	1/16	1/16	1/8	1/8	1/8	1/8
2	1/32	1/32	1/64	1/64	0	2/64

Find the marginal distributions, conditional distribution of X given $Y = 1$ and conditional distribution of Y given $X = 0$.

The marginal distributions are given in the table below.

X \ Y	1	2	3	4	5	6	$P(X=x)$
0	0	0	$1/32$	$2/32$	$2/32$	$3/32$	$8/32$
1	$1/16$	$1/16$	$1/8$	$1/8$	$1/8$	$1/8$	$10/16$
2	$1/32$	$1/32$	$1/64$	$1/64$	0	$2/64$	$8/64$
$P(Y=y)$	$3/32$	$3/32$	$11/64$	$13/64$	$6/32$	$16/64$	1

Conditional distribution of X given $Y = 1$.

$$P(X=0/Y=1) = \frac{P(X=0, Y=1)}{P(Y=1)} = 0$$

$$P(X=1/Y=1) = \frac{P(X=1, Y=1)}{P(Y=1)} = \frac{1/16}{3/32} = 2/3$$

$$P(X=2/Y=1) = \frac{P(X=2, Y=1)}{P(Y=1)} = \frac{1/32}{3/32} = 1/3$$

Conditional distribution of Y given $X = 0$

$$P(Y=1/X=0) = \frac{P(X=0, Y=1)}{P(X=0)} = 0$$

$$P(Y=2/X=0) = \frac{P(X=0, Y=2)}{P(X=0)} = 0$$

$$P(Y=3/X=0) = \frac{P(X=0, Y=3)}{P(X=0)} = \frac{1/32}{8/32} = 1/8$$

$$P(Y=4/X=0) = \frac{P(X=0, Y=4)}{P(X=0)} = \frac{2/32}{8/32} = 2/8$$

$$P(y=5/x=0) = \frac{P(x=0, y=5)}{P(x=0)} = \frac{2/32}{8/32} = 2/8$$

$$P(y=6/x=0) = \frac{P(x=0, y=6)}{P(x=0)} = \frac{3/32}{8/32} = 3/8$$

(3) The joint pmf of (x,y) is given by $p(x,y) = k(2x+3y)$; $x = 0,1,2$; $y = 1,2,3$. Find all the marginal and conditional probability distributions. Also find the probability distribution of $(x+y)$.

The marginal distributions are given in the table below.

X \ Y	1	2	3	$P(X=x)$
0	3K	6K	9K	18K
1	5K	8K	11K	24K
2	7K	10K	13K	30K
$P(Y=y)$	15K	24K	33K	72K

$$\sum_j \sum_i P(x_i, y_j) = 1 \Rightarrow 72K = 1$$

$$\boxed{K = 1/72}$$

X \ Y	1	2	3	$P(X=x)$
0	$3/72$	$6/72$	$9/72$	$18/72$
1	$5/72$	$8/72$	$11/72$	$24/72$
2	$7/72$	$10/72$	$13/72$	$30/72$
$P(Y=y)$	$15/72$	$24/72$	$33/72$	1

Conditional distribution of X given $Y = y$

when $Y = 1$

$$P[X=0/Y=1] = \frac{P[X=0, Y=1]}{P[Y=1]} = \frac{3/72}{15/72} = \frac{1}{5}$$

$$P[X=1/Y=1] = \frac{P[X=1, Y=1]}{P[Y=1]} = \frac{5/72}{15/72} = \frac{1}{3}$$

$$P[X=2/Y=1] = \frac{P[X=2, Y=1]}{P[Y=1]} = \frac{7/72}{15/72} = \frac{7}{15}$$

when $Y=2$

$$P[X=0/Y=2] = \frac{P[X=0, Y=2]}{P[Y=2]} = \frac{6/72}{24/72} = \frac{1}{4}$$

$$P[X=1/Y=2] = \frac{P[X=1, Y=2]}{P[Y=2]} = \frac{8/72}{24/72} = \frac{1}{3}$$

$$P[X=2/Y=2] = \frac{P[X=2, Y=2]}{P[Y=2]} = \frac{10/72}{24/72} = \frac{5}{12}$$

when $y = 3$

$$P[x=0/y=3] = \frac{P[x=0, y=3]}{P[y=3]} = \frac{9/72}{33/72} = \frac{3}{11}$$

$$P[x=1/y=3] = \frac{P[x=1, y=3]}{P[y=3]} = \frac{11/72}{33/72} = \frac{1}{3}$$

$$P[x=2/y=3] = \frac{P[x=2, y=3]}{P[y=3]} = \frac{13/72}{33/72} = \frac{13}{33}$$

conditional distribution of y given $x = x$

when $x = 0$

$$P[y=1/x=0] = \frac{P[x=0, y=1]}{P[x=0]} = \frac{3/72}{18/72} = \frac{1}{6}$$

$$P[y=2/x=0] = \frac{P[x=0, y=2]}{P[x=0]} = \frac{6/72}{18/72} = 1/3$$

$$P[y=3/x=0] = \frac{P[x=0, y=3]}{P[x=0]} = \frac{9/72}{18/72} = \frac{1}{2}$$

<u>when $x = 1$</u>

$$P[y=1/x=1] = \frac{P[x=1, y=1]}{P[x=1]} = \frac{5/72}{24/72} = \frac{5}{24}$$

$$P[y=2/x=1] = \frac{P[x=1, y=2]}{P[x=1]} = \frac{8/72}{24/72} = \frac{1}{3}$$

$$P[y=3/x=1] = \frac{P[x=1, y=3]}{P[x=1]} = \frac{11/72}{24/72} = \frac{11}{24}$$

<u>when $x = 2$</u>

$$P[y=1/x=2] = \frac{P[x=2, y=1]}{P[x=2]} = \frac{7/72}{30/72} = 7/30$$

$$P[y=2/x=2] = \frac{P[x=2, y=2]}{P[x=2]} = \frac{10/72}{30/72} = 1/3$$

$$P[y=3/x=2] = \frac{P[x=2, y=3]}{P[x=2]} = \frac{13/72}{30/72} = \frac{13}{30}$$

Probability distribution of $(x+y)$	
$(x+y)$	P
1 $P(0,1)$	$\frac{3}{72}$
2 $P(0,2)+P(1,1)$	$\frac{6}{72}+\frac{5}{72}=\frac{11}{72}$
3 $P(0,3)+P(1,2)+P(2,1)$	$\frac{9}{72}+\frac{8}{72}+\frac{7}{72}=\frac{24}{72}$
4 $P(1,3)+P(2,2)$	$\frac{11}{72}+\frac{10}{72}=\frac{21}{72}$
5 $P(2,3)$	$\frac{13}{72}$.
Total	1